THE JOURNAL OF ENGINEERING EDUCATION

lished by THE SOCIETY FOR THE PROMOTION OF ENGINEERING EDUCATION

7 SERIES
. XXII., NO. 4 | DECEMBER, 1931 | Subscription Price $3.00 per year
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THE JOURNAL OF ENGINEERING EDUCATION

Published by the Society for the Promotion of Engineering Education

Published under the Supervision of the Publication Committee.
H. S. Evans, H. S. Boardman, F. L. Bishop, Editor, University of Pittsburgh.
Published at Prince and Lemon Sts., Lancaster, Pa.

Entered as second-class matter at the Post Office at Lancaster, Pa., under the Act of March 3, 1870.

New Series
Vol. XXII, No. 4. DECEMBER, 1931 Subscription Price, $3.00 per year. Single Copies, 50 cents.

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RESEARCH IN INDUSTRY *

By DAVID E. ROSS

Chairman, Board of Trustees of Purdue University

This subject rather alarmed me when Dean Potter discussed with me what I was to talk about. I knew there would be men from great organizations here carrying on great research work, and then I thought of that story of an Englishman who came to this country. He went back to England and told the people how wonderfully our industry was mechanized. He said, "It doesn't seem to disturb them any. I was talking to a business man one day and he looked at his watch and said, 'I will have to go. I just have five minutes to *make* the streetcar.'" Dean Potter told me I would just have five minutes.

You know, back in the days of the carbon steel tools the engineer developed machine tools that would go to the capacity of that carbon steel tool, and then some man in a back laboratory brought out tungsten. That meant the redesigning of all those machine tools. This engineer went ahead with those machine tools. He made them multiple spindle and multiple tooled. He timed them and synchronized them until he bragged how many men this one displaced, and how many men that one displaced. He didn't say what became of them. It was a wonderful achievement and any industry that wanted to keep up had to use the new development.

About the time they had them going, along comes another man in a laboratory and said "Carboloy." They all knew what that meant. They all know what that means now better than when they first heard of tungsten. Only last week I sat with a group of men and something new was shown. A big industrialist turned to the man interested and said, "How much does that make obsolete?"

He said, "For me, several hundred thousand dollars." But he did not say how many people trained to operate that $250,000 would probably never have as good a place again.

I am citing these to you in this organization for a very definite reason. As cycles of education go, just a short time ago we were an experiment, technical education, and classic and academic schools rather looked down on this type of education. Our athletes were called boiler makers. When I was a student that was a term of derision of other schools. We took it and kept it, and now we have added riveters to it.

* Presented at the 39th annual meeting, S. P. E. E., Purdue University, June 17–19, 1931.

You work in a virgin field, and you have upset a lot of traditions. You have created a large number of new variables for the economist. He does not know what to do with them. A great industry that years ago the economists would figure was a basic has been synthesized, and then comes another one, and another one. All their obligations were rendered obsolute and the great groups working for them are obsolete.

My suggestion is that you take your people who are leaders and bring them back and give them supertraining in leadership, because I believe as the others have all failed this is the group that will have to remedy the wreckage they have created.

The obsoleting of people, putting them where they never again can take the places they had before, and the obsoleting of investment is bad enough. Your creating a technological unemployment is bad enough, and some day if we do not correct this the mass may correct it for us. It is your problem. I have every belief that you will solve it.

It was said yesterday or the day before that any man was foolish to predict, so I am going to be foolish. I am going to predict that when you get that solution it will be an education of how to spend leisure, whether leisure will be a cultural leisure or not; whether leisure will be vicious or not.

Can you solve the problem of distribution as you have solved the problem of production? What will people do with their extra time? If it is a cultural leisure, there is hope for this country.

PURDUE UNIVERSITY CLASSIFICATION OF THE ENGINEERING ALUMNI

CLASSES OF 1920 TO 1930, INCLUSIVE

BY A. A. POTTER AND J. E. WALTERS

The engineering alumni of Purdue University (classes 1920–1930) were classified in Tables I, II, III and IV according to the Line Type of Employment, Functional Type of Employment, Location, and Division of Engineering.

The line type of employment (Table I) was subdivided into the various forms of occupations, devised in the Personnel Office of the Engineering Schools of Purdue University.

The classifications given in the *Report of the Investigation of the Society for the Promotion of Engineering Education (1923–1929)* were used for the functional type of employment (Table II) and for the divisions of engineering (Table IV).

In regard to location (Table III) general usage was followed. Since Purdue University is a state institution a division of alumni in Indiana was added.

The last Alumni Directory provided the source of information for the classes of 1920 to 1928. Information for the classes of 1929 and 1930 was secured from the Alumni Office.

Due to the fact that some of the alumni have supplied only a very brief account of their occupation, it was difficult to classify these, especially with respect to their functional type of employment.

TABLE I

Engineering Alumni Classified as to Line Type of Employment

Line Type of Employment	1920		1921		1922		1923		1924		1925		1926		1927		1928		1929		1930		Total Ave.	
	No.	%	No.	%	No.	%	No.	%	No.	%	No.	%	No.	%	No.	%	No.	%	No.	%	No.	%	No.	%
1 Mfg.	74	43.5	75	34.2	117	42.4	139	41.7	85	33.1	118	38.9	106	37.5	113	44.2	128	45.5	134	41.3	164	42.1	1253	40.5
Steel, Iron	5	2.9	8	3.5	12	4.3	12	3.6	11	4.3	14	4.6	15	5.3	13	5.2	12	4.3	11	3.4	17	4.4	130	4.2
Electrical	10	5.9	16	7.3	27	9.8	34	10.2	22	8.6	33	10.8	41	14.5	29	11.3	45	16.0	40	12.3	34	8.7	331	10.7
Oil, Gas	2	1.2	5	2.3	8	2.9	9	2.7	6	2.3	10	3.4	6	2.1	8	3.2	9	3.2	11	3.4	13	3.4	87	2.8
Rubber, Glass	5	2.9	3	1.4	4	1.4	2	.6	4	1.6	3	1.0	3	1.1	11	4.3	8	2.8	10	3.1	9	2.3	62	2.0
Spl. Mach.	25	14.7	17	7.8	32	11.6	45	13.5	18	7.0	28	9.2	21	7.4	24	9.3	30	10.7	34	10.5	50	12.8	324	10.5
Miscl.	27	15.9	26	11.9	34	12.4	37	11.1	24	9.3	30	9.9	20	7.1	28	10.9	24	8.5	28	8.6	41	10.5	319	10.3
2 Transportation	9	5.3	15	6.8	22	8.0	30	8.9	31	12.2	32	10.5	49	17.3	39	15.2	46	14.5	62	19.1	56	14.1	391	12.6
Railways	1	.6	4	1 8	14	5.1	14	4.3	18	7.1	20	6.6	27	9.5	19	7.4	18	6.4	24	7.4	23	5.9	182	5.9
Highways	6	3.5	6	2.7	2	.7	10	2.8	9	3.5	4	1.3	13	4.6	11	4.3	20	7.2	15	4.6	24	6.2	120	3.9
Automobile	2	1.2	4	1.8	5	1.8	6	1.8	3	1.2	3	1.0	5	1.8	9	3.5	3	1.1	9	2.8	2	.5	51	1.6
Aviation	0	0.0	1	.5	1	.4	0	0.0	1	.4	5	1.6	4	1.4	0	0.0	5	1.8	14	4.3	7	1.8	38	1.2
3 Power	12	7.1	17	7.8	22	8.0	24	7.2	29	11.3	34	11.2	25	8.8	19	7.4	20	7.2	26	8.0	19	4.9	247	8.0
4 Communication	7	4.1	19	8.7	14	5.1	14	4.3	7	2.6	8	2.6	13	4.6	11	4.4	13	4.6	39	12.1	32	8.2	177	5.7
Telephone	7	4.1	19	8.7	13	4.7	14	4.3	7	2.6	7	2.3	13	4.6	10	4.0	11	3.9	32	9.9	27	6.9	160	5.2
Radio	0	0.0	0	0.0	1	.4	0	0.0	0	0.0	1	.3	0	0.0	1	.4	2	.7	7	2.2	5	1.3	17	0.5
5 Public Works	7	4.2	8	3.7	10	3.6	17	4.9	10	3.8	20	6.5	8	2.8	19	7.1	9	3.2	15	4.6	14	3.7	137	4.5
City	4	2.4	5	2.3	4	1.4	10	2.8	5	1.9	6	1.9	5	1.8	5	1.9	4	1.4	5	1.5	5	1.3	58	1.9
State	1	.6	3	1.4	3	1.1	4	1.2	1	.4	7	2.3	1	.3	7	2.6	3	1.1	2	.6	1	.3	33	1.1
U. S.	2	1.2	0	0.0	3	1.1	3	.9	4	1.5	7	2.3	2	.7	7	2.6	2	.7	8	2.5	8	2.1	46	1.5
6 Contracting	4	2.4	6	2.7	9	3.3	16	4.8	6	2.3	12	3.9	13	4.6	7	2.6	8	2.8	5	1.5	20	5.1	106	3.4
7 Education	11	6.4	13	5.8	10	3.7	12	3.4	10	4.0	11	3.7	7	2.5	9	3.5	9	3.2	9	2.8	17	4.3	118	3.8
High School	6	3.5	5	2.3	1	.4	2	.6	2	.8	1	.3	0	0.0	0	0.0	0	0.0	2	.6	0	0.0	19	0.6
University	5	2.9	8	3.5	9	3.3	10	2.8	8	3.2	10	3.4	7	2.5	9	3.5	9	3.2	7	2.2	17	4.3	99	3.2
8 Mining, Arch.	0	0.0	1	.5	3	1.1	2	.6	4	1.6	3	1.0	2	.7	0	0.0	0	0.0	0	0.0	1	.3	16	0.5
9 Non Engng.	7	4.1	14	6.4	9	3.3	13	3.9	5	1.9	10	3.3	4	1.4	4	1.6	6	2.1	3	.9	3	.8	78	2.5
10 Miscl.	4	2.3	1	.5	6	2.2	5	1.4	6	2.3	11	3.6	9	3.2	4	1.6	7	2.5	10	3.1	16	4.1	79	2.5
11 Unknown	31	18.2	40	18.3	50	18.2	59	17.5	61	23.7	45	14.8	47	16.6	32	12.4	34	12.1	20	6.2	44	11.3	463	15.0
12 Deceased	4	2.4	10	4.6	3	1.1	5	1.4	3	1.2	0	0.0	0	0.0	0	0.0	1	.3	1	.4	3	.8	30	1.0
Total	170		219		275		336		257		304		283		257		281		324		389		3095	

12 Deceased	4	2.4	10	4.6	3	1.1	5	1.4	3	1.2	0	0.0	0	0.0	0	0.0	1	.3	1	.4	3	.8	30	1.0
Total	170		219		275		336		257		304		283		257		281		324		389		3095	

TABLE II

Engineering Alumni Classified as to Functional Type of Employment

Functional Type of Employment	1920		1921		1922		1923		1924		1925		1926		1927		1928		1929		1930		Total Ave.	
	No.	%	No.	%	No.	%	No.	%	No.	%	No.	%	No.	%	No.	%	No.	%	No.	%	No.	%	No.	%
1 Consulting Engng.	1	.6	2	.9	1	.4	1	.3	1	.4	4	1.3	2	.7	3	.8	0	0.0	0	0.0	0	0.0	14	.5
2 Design, Estimating, and Drafting	7	4.1	6	2.8	11	4.0	20	5.9	15	5.8	19	6.3	23	8.1	18	7.1	15	5.3	20	6.2	18	4.6	172	5.6
3 Const., Opr., Testing—Asst. Engrs.	14	8.3	25	11.2	36	13.1	56	16.8	39	14.8	60	19.8	65	23.1	69	26.8	104	36.9	122	37.5	137	35.2	727	23.5
Construction	4	2.4	2	.9	4	1.4	9	2.7	4	1.6	5	1.6	7	2.5	5	1.9	7	2.5	4	1.2	8	2.1	59	1.9
Operation	2	1.2	6	2.7	6	2.2	9	2.7	5	1.8	8	2.6	10	3.5	16	6.2	13	4.6	18	5.5	10	2.6	103	3.3
Maintenance	0	0.0	2	.9	3	1.1	4	1.2	3	1.2	6	1.9	6	2.1	7	2.6	14	5.0	10	3.1	13	3.4	68	2.2
Testing	0	0.0	2	.9	2	.7	7	2.2	5	1.8	8	2.6	8	2.8	3	1.2	18	6.4	8	2.5	6	1.5	67	2.2
Inspection	1	.6	3	1.4	3	1.1	6	1.8	5	1.8	11	3.6	5	1.8	4	1.6	9	3.2	8	2.5	10	2.6	65	2.1
Production	1	.6	2	.9	5	1.8	5	1.4	2	.8	7	2.3	9	3.2	8	3.2	8	2.8	4	1.2	4	1.0	55	1.8
Asst. Engrs.	6	3.5	8	3.5	13	4.8	16	4.8	15	5.8	15	4.9	20	7.1	26	10.1	35	12.4	70	21.5	86	22.0	310	10.0
4 Teaching and Research	14	8.2	25	11.4	26	9.4	24	7.2	17	6.6	29	9.5	20	7.1	30	11.7	18	6.4	34	10.5	45	11.6	282	9.1
5 Mgt., Admin.	72	42.4	82	37.5	106	38.6	120	35.7	82	32.2	109	35.8	80	28.2	81	31.5	66	23.6	87	26.9	83	21.3	968	31.1
Chief Engr.	3	1.8	7	3.2	4	1.4	4	1.2	2	.8	2	.7	0	0.0	2	.8	0	0.0	1	.4	0	0.0	25	0.8
Engineer	24	14.1	44	20.1	62	22.6	70	20.8	52	20.2	71	23.3	55	19.4	61	23.7	54	19.3	72	22.2	71	18.2	636	20.4
Mgr., Trea. and Pres.	30	17.6	23	10.5	18	6.6	18	5.5	13	5.2	22	7.2	9	3.2	4	1.6	7	2.5	2	.6	7	1.8	153	4.9
Supt.	12	7.1	6	2.8	17	6.2	23	6.8	12	4.8	10	3.3	12	4.2	9	3.5	2	.7	9	2.8	5	1.3	117	3.8
Foremen and Employers	3	1.8	2	.9	5	1.8	5	1.4	3	1.2	4	1.3	4	1.4	5	1.9	3	1.1	3	.9	0	0.0	37	1.2
6 Sales	20	11.7	17	7.8	32	11.6	32	9.5	29	11.3	21	6.9	32	11.3	15	5.8	23	8.2	25	7.7	28	7.2	274	8.9
7 Clerical and Miscl.	7	4.1	12	5.5	10	3.6	19	5.7	10	4.0	17	5.6	14	4.9	10	4.0	20	7.2	15	4.6	31	8.0	165	5.3
8 Unknown	31	18.2	40	18.3	50	18.2	59	17.5	61	23.7	45	14.8	47	16.6	32	12.3	34	12.1	20	6.2	44	11.3	463	15.0
9 Deceased	4	2.4	10	4.6	3	1.1	5	1.4	3	1.2	0	0.0	0	0.0	0	0.0	1	.3	1	.4	3	.8	30	1.0
Total	170		219		275		336		257		304		283		257		281		324		389		3095	

TABLE III

Engineering Alumni Classified as to Location

Location	1920		1921		1922		1923		1924		1925		1926		1927		1928		1929		1930		Total Ave.	
	No.	%	No.	%	No.	%	No.	%	No.	%	No.	%	No.	%	No.	%	No.	%	No.	%	No.	%	No.	%
Indiana	58	35.	66	31.6	99	36.4	101	30.5	83	32.7	89	29.4	99	34.9	84	32.6	97	34.6	102	31.5	116	30.	994	32.4
Central * States	50	30.1	70	33.5	87	32.	125	37.8	92	36.2	108	35.9	104	36.6	98	38.	108	38.6	110	34.	137	35.5	1089	35.5
Eastern States	24	14.5	32	15.3	37	13.6	44	13.3	26	10.2	45	13.9	39	13.7	40	15.5	44	15.7	63	19.4	86	22.3	480	15.6
Southern States	9	5.4	19	9.1	21	7.8	30	9.1	19	7.5	25	8.7	13	4.6	15	5.8	10	3.6	19	5.9	9	2.3	189	6.2
Western States	12	7.2	8	3.8	11	4.	17	5.1	18	7.1	12	4.2	11	3.9	6	2.3	6	2.1	11	3.4	25	6.5	137	4.5
Far West States	11	6.6	5	2.4	12	4.4	8	2.4	4	1.6	3	1.	6	2.1	3	1.2	5	1.8	4	1.2	1	.3	62	2.
Foreign	2	1.2	9	4.3	5	1.8	6	1.8	12	4.7	21	6.8	12	4.2	12	4.6	10	3.6	15	4.6	12	3.1	116	3.8
Total	166		209		272		331		254		303		284		258		280		324		386		3067	

* Not including Indiana.

TABLE IV

Engineering Alumni Classified as to Division of Engineering

Division of Engineering	1920		1921		1922		1923		1924		1925		1926		1927		1928		1929		1930		Total Ave.		S.P.E.E.*
	No.	%	No.	%	No.	%	No.	%	No.	%	No.	%	No.	%	No.	%	No.	%	No.	%	No.	%	No.	%	%
Same as Course	86	63.7	106	59.2	141	63.5	175	64.3	132	68.4	174	67.2	174	73.7	161	71.6	187	76.	224	74.2	248	72.5	1808	69.2	61.8
Closely associated	13	9.7	31	17.3	32	14.4	35	12.9	22	11.4	36	13.9	23	9.7	31	13.8	26	10.6	39	12.9	37	10.8	325	12.5	7.4
Unassoc	18	13.3	20	11.2	28	12.6	36	13.2	23	11.9	31	12.	24	10.2	23	10.2	21	8.5	26	8.6	33	9.6	283	10.8	16.0
Non Engng	18	13.3	22	12.3	21	9.5	26	9.6	16	8.3	18	6.9	15	6.4	10	4.	12	4.9	13	4.3	24	7.1	195	7.5	14.8
Total	135		179		222		272		193		259		236		225		246		302		342		2611		

* From Vol. 1, Report of the Investigation of Engineering Education, 1923–29—Society for the Promotion of Engineering Education.

Note: Approximately 15 per cent of the alumni were unknown as to their division of Engineering. The above classification is made on the basis of those known.

HISTORY OF MATHEMATICS *

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Prefatory Note

An attempt is here made to give indications of the development of mathematics before the nineteenth century, and to refer briefly to some developments of the nineteenth century in connection with topics usually discussed at undergraduate colleges. Chinese and Japanese mathematics are not considered in such a skeleton survey, and the reference to mathematics of the Hindus is brief.

It was felt that consideration of only what was most important would not be desirable in the survey, partly because many of the most important things can not, in a few sentences, be made intelligible to one wholly unfamiliar with the subject, and partly because the quotation of complicated results in a lecture is not possible. Hence many minor items in the development of the science have been introduced in order that interest may be sustained through knowing the historical setting for certain simply stated, or well-known, results.

While an endeavor has been made to present the results of the latest investigations, no claim for originality can be made. Indeed phrases, quotations, and adaptations from many sources have been used constantly. In the Literature List all such sources are indicated and in the text many references are given to them, that the reader may if he choose inform himself more fully concerning the matter discussed. The sources indicated are almost wholly in English, and for our outline survey, they serve fairly well. A scholarly grasp of any field would, however, necessitate the consultation of literature in other languages, but especially in German and many sources of this kind have been consulted.

Synopsis

Following Smith (see Literature List, no. 4) I have used such an abbreviation as -1850 for 1850 B.C., in the synopsis which follows, and have associated with the name of each mathematician a single

* Two lectures delivered before the Mathematics Session of the Summer School for Engineering Teachers, University of Minnesota, August 31 and September 1, 1931.

definite date indicating a year about when he flourished or when he published some outstanding work.

I. History of Mathematics before 1600 A.D.

A. Babylonian and Egyptian Mathematics 3500 B.C. to 600 B.C.

Babylonian: Geometry, solution of quadratic equations
Egyptian: Moscow papyrus, −1850; Rhind papyrus, −1650; Arithmetic; geometry; great pyramid, −2900; obelisks; Nile survey

B. Greek Mathematics 600 B.C. to 500 A.D.

Thales (−600): Geometry, solar eclipse of −585
Pythagoras (−540): Geometry, arithmetic, music, astronomy
Menaechmus (−350): Conic sections
Famous problems (−450):
(*a*) Trisection of an angle: Hippias of Elis (−425) with quadratrix, Nicomedes (−180) with conchoid
(*b*) Squaring the circle: Dinostratus (−350) with quadratrix, Archimedes (−225) with his spiral
(*c*) Duplication of the cube: Hippocrates of Chios (−460) with two mean proportionals, Archytas (−400) with surfaces, Diocles (−180) with cissoid
Euclid (−300): *Elements*, perfect numbers
Aristarchus of Samos (−287): Copernican system
Archimedes (−225): Geometry, arithmetic, mechanics, hydrostatics
Eratosthenes (−230): Size of the earth, sieve
Apollonius of Perga (−225): *Conic Sections*, *Plane Loci*, *Tangencies*
Hipparchus (−140): Trigonometry, precession of the equinoxes
Ptolemy of Alexandria (150, *i.e.* 150 A.D.): Trigonometry, stereographic projection
Heron of Alexandria (180): *Pneumatica*, cube root, volume of torus
Diophantus of Alexandria (275): Indeterminate equations
Pappus of Alexandria (300): *Mathematical Collections*

C. Hindu, Arabic, Persian Mathematics 500 to 1200

Brahmagupta (630): Area of an inscribed quadrilateral
Bhāskara (1150): Solution of "Pell's equation," linear indeterminate equations
Hindu-Arabic numerals (−300 − +876)
al-Khowârizmi (820): *'ilm al-jabr wa'l muqabalah*, algebra
Abû'l Wefâ (980): Geometrical constructions with ruler, and compasses having a fixed opening
Omar Khayyàm (1100): Solutions of cubic equations, calendar

D. European Mathematics 1200 to 1600

Leonardo of Pisa (Fibonacci) (1202): *Liber Abaci*, *Liber Quadratorum*, *Practica Geometriae*
Johann Müller (Regiomontanus) (1470): First systematic work on trigonometry
First printed arithmetic (1478) with date
First printed edition of Euclid's *Elements* (1482)
Scipio del Ferro and Antonio Fior (1506): Cubic equation
Nicolo of Brescia (Tartaglia) (1545): Cubic equation

Girolamo Cardano (1545): Algebra
Ferrari (1545): Biquadratic equation
Georg Rhaeticus (1550): Tables of natural trigonometric functions
Nicolaus Copernicus (1520): System of the universe
François Viète (1580): Trigonometric expansions, equations, π as infinite product, notations
First mathematical book published in the New World (1556)

II. History of Mathematics after 1600

A. The Seventeenth Century

John Napier (1614): Logarithms, trigonometry, Nap. $\log x = 10^7 \log (1/e)(x/10^7)$
Johann Kepler (1610): Laws for motion of planets, geometry
Galileo Galilei (1600): Falling bodies, projectiles, cycloid
Thomas Harriot (1600): Algebra
William Oughtred (1630): *Clavis Mathematicae*, algebra, slide rule, notations
Gérard Desargues (1640): Projective geometry, treatise on singing
Étienne Pascal: Limaçon
Blaise Pascal (1650): Conics, computing machine, cycloid, theory of probability
George Mohr (1672): Constructions with compasses only
René Descartes (1637): Analytic geometry, equations, $F + V = E + 2$, folium
Pierre de Fermat (1635): Theory of numbers, maxima and minima, theory of probabilities
Bonaventura Cavalieri (1635): Indivisibles
Christiaan Huygens (1670): Circle quadrature, theory of probabilities, evolutes, pendulum clocks, catenary
John Wallis (1650): Algebra, imaginary numbers, length of curves, areas, π as infinite product
Isaac Newton (1680): Fluxions, dynamics, hydrostatics, hydrodynamics, gravitation, cubic curves, series, numerical equations, imaginary roots, challenges
Gottfried Wilhelm, Freiherr von Leibniz (1682): Newton controversy, calculus, determinants, polynomial expansions, notations
Jacques Bernoulli (1690): Isochronous curves, Euler's spiral, logarithmic spiral, calculus of probability

B. Eighteenth Century

Abraham DeMoivre (1720): Actuarial mathematics, theory of probabilities, trigonometry with complex quantities
Brook Taylor (1720): Expansion in series, finite differences, vibrating string
Colin Maclaurin (1740): Fluxions, organic description of curves
Léonard Euler (1750): Publications, notations, $e^{i\pi} + 1 = 0$, Euler's line, biquadratic equation, indicatrix, vibrating string, calculus of variations, Euler's spiral, beta and gamma functions, Euler's constant
Joseph Louis Lagrange (1780): Calculus of variations, differential equations, mechanics, calculus foundations, theory of numbers, numerical equations
Johann Heinrich Lambert (1770): *Freye perspective*, parabola property, π irrational, hyperbolic functions, map projection
Gaspard Monge (1795): Descriptive geometry, differential geometry

C. Nineteenth Century and Later

Pierre Simon, Marquis de Laplace (1805): Astronomy, celestial mechanics, probability, differential equations
Adrien Marie Legendre (1805): Theory of numbers, elliptic functions and integrals, exercises in integral calculus, geometry
Non-Euclidean geometry: Bolyai and Lobachevsky (1825)
Karl Friedrich Gauss (1820): Polygon constructions, theory of numbers, differential geometry, algebraic equations, etc.
Calculating prodigies, Zacharias Dase
Projective and modern geometry: Victor Poncelet (1830) and Jakob Steiner (1840), duality, Charles J. Brianchon (1806)
Trisection of an angle: Duplication of cube proved impossible (1837)
Inversion: Steiner (1824), Quetelet (1825), linkages
Algebraic solution of the general quintic equation impossible: Niels Henrik Abel (1824)
Approximating roots of numerical equations: Newton (1669); Ruffini (1804), Horner (1819)
Fourier's theorem for limit number roots in an interval (1820)
Sturm's theorem for exact number of roots in an interval (1829)
Cauchy's theorem for number of real or imaginary roots within a given contour
Convergence of series: Newton, Gauss, Cauchy; Fourier Series (1812–22)
Vector analysis: Gibbs (1881–84), Heaviside (1891)
Quaternions: W. R. Hamilton (1853)
Determinants
Mathematical Tables: Duffield (1897), Lehmer (1910–1914), Becker and Van Orstrand (1909), Pernot and Woods (1915–1918), Andoyer (1915–1918)
Guides to Tables: DeMorgan (1861) Glaisher and Cayley (1873–1883), v. Mises (1928), Henderson (1926)
Surveys of pure and applied mathematics: E. W. Brown (1923), R. S. Woodward (1900), J. W. Nicholson (1922)

HISTORY OF MATHEMATICS BEFORE THE SEVENTEENTH CENTURY

A. Babylonian and Egyptian Mathematics 3500 B.C.–600 B.C.*

The period covered by this first lecture on the history of mathematics will be about 5000 years; for we shall begin by noting that in one of the great museums at Oxford is a royal mace of 3500 B.C. on which there is a record of 120,000 prisoners, of 400,000 captive oxen, and of 1,422,000 captive goats. These numbers, written in Egyptian hieroglyphs, prove that already in this ancient time not only was the decimal system of numeration, but also a method for writing very large numbers, thoroughly established. It is interesting to speculate

* Such a number as this is a reference to the corresponding item in the "Literature List," where more details concerning the topic referred to may be found. See also nos. 7–11.

on how many thousand years earlier must have been *the beginnings*, when great communities, of a millenium later, with a highly developed social order, calling for the frequent use of mathematics of taxation, barter, interest, and of large numbers, were replaced by small groups of primitive people for whose simple needs such number words as "one," "two," and "heap" would wholly suffice.[4]

Any general primitive notions of this kind probably long antedated the execution of the recently discovered finest art-work of primitive man, reputed to be from 25,000 to 50,000 years old.[5] It is a sculpture, of a white rhinoceros with a swarm of attendant tick-birds, hammered into a slab of basaltic rock, and showing extraordinary power of form, line, and perspective—elements intimately allied to things mathematical.

But there were other people beside the Egyptians who contributed notably to early mathematics. I refer to the non-Semitic Sumerians who lived just north of the Persian Gulf and south of the Semitic Akkadians, and who for at least a thousand years prior to 2500 B.C. were generally predominant in Babylonia, but were absorbed in a larger political group by about 2000 B.C. One of the greatest of the Sumerian inventions was the adoption of cuneiform script; notable engineering works of the Babylonians, by means of which marshes were drained and the overflow of rivers regulated by canals, went back to Sumerian times, as also a considerable part of their religion, their law, and their system of mathematics.

An extraordinary number of tablets show that the Sumerian merchant of 2500 B.C. was familiar with such things as weights and measures, bills, receipts, notes, and accounts. Long before coins were in use (7th century B.C.) it was the common custom to pay interest for the loan of produce, or of a certain weight of a precious metal. Sumerian tablets indicate that the rate of interest varied from 20 to 30 per cent, the higher rate being charged for produce.

Sumerian or Babylonian arithmetic was essentially sexagesimal and while a special symbol for 10 was constantly used it occupied a subordinate position; there were no symbols for 100, which was thought of as $60 + 40$, and for 1,000, thought of as $16 \cdot 60 + 40$. In this latter case the number was written simply $16 \cdot 40$. So also $31 \cdot 6 \cdot 15$ might equal 111,975. Their symbol for unity was also the symbol for any one of the numbers 60^n where n is a positive or negative integer. Thus $31 \cdot 6 \cdot 15$ might mean not only 111,975, but also 6,718,500 or 1,866 1/4 or 31 1/10 1/240, according as the 15 is taken as multiplied by 1, or 60, or 1/60, or 1/3600. The uncertainty in this regard is one of the great difficulties in interpreting Babylonian mathematical texts. Another uncertainty was introduced through the fact that a blank space sometimes meant zero, so that $10 \cdot \cdot 7$ might stand for 36,007. No special symbol for zero was used by the Babylonian till near the beginning of the Christian era.

In the field of geometry the Babylonian of 2200–2000 B.C. used the following results, in concrete cases from which we have to infer that they were familiar with the general rules:

1. The area of a rectangle is the product of the lengths of two adjacent sides.
2. The area of a right triangle is equal to one-half the product of the lengths of the sides about the right angle.
3. The area of a trapezium with one side perpendicular to the parallel sides is one-half the product of the length of this perpendicular and the sum of the lengths of the parallel sides.
4. The length of the diameter of a circle is one-third of its circumference ($\pi = 3$).
5. The area of a circle is 1/12 of the square of its circumference (correct for $\pi = 3$).
6. The length of a side of a right angled triangle is equal to the square root of the difference of the squares of the lengths of the hypotenuse and of the other side (at least in the case of a right triangle whose sides are in the ratio of 3, 4, and 5).

This last result, which is equivalent to the Pythagorean theorem, occurs in a problem to find the length of a chord of a circle when the height of its arc, 2, and the circumference, 60, of the circle are known. The tablets containing this problem, and others of a very extraordinary nature, have been interpreted and published only within the past three years. Some of the discussion of geometrical problems is more akin to algebra than to geometry; the steps taken seem indubitably to lead to the solution of systems of linear simultaneous equations, or to the solution of quadratic equations. Moreover the Babylonian of 2000 B.C. evidently knew our formula for the solution of a quadratic equation with the positive sign before the radical. A number of problems could be cited to prove this; let us consider one of them [15] on a tablet in Strassburg dating from about 2000B.C.

'An area $[A]$ (consisting of) the sum of two squares $[x^2 + y^2]$ (is) $16\cdot40$ $[= 1000]$. (One) square $[y]$ (is) $\frac{2}{3}\left[\frac{\alpha}{\beta}\right]$ of (the other) square, while $10[d]$ is subtracted from the lesser square. Calculate the squares $[x$ and $y]$.'

$$x^2 + y^2 = A, \qquad y = \frac{\alpha}{\beta}x - d.$$

If $\zeta = x/\beta$, we get a quadratic equation:

$$\zeta^2 - \frac{2d\alpha}{\alpha^2 + \beta^2}\zeta = \frac{A - d^2}{\alpha^2 + \beta^2},$$

the solution being

$$\zeta = \frac{1}{\alpha^2 + \beta^2}[d\alpha \pm \sqrt{\{d^2\alpha^2 + (\alpha^2 + \beta^2)(A - d^2)\}}].$$

Bearing in mind that, in the particular case, $A = 1000$, $d = 10$, $\alpha = 40$, $\beta = 60$, let us now compare with this the working given in the text.

"You proceed thus:	
Square 10: this gives [1·40] 100; subtract [1·40] 100 from [16·40] 1000: this gives [15·0] 900.	$A - d^2 = 900.$
Square [1·0] 60: this gives [1·0·0] 3600; 40^2 is [26·40] 1600: 3600 + 1600 = 5200.	$\alpha^2 + \beta^2 = 5200.$
Multiply 5200 by 900: this gives 4680000.	$(\alpha^2 + \beta^2)(A - d^2) = 4680000.$
Multiply 40 by 10: this gives 400.	$\alpha d = 400.$
Square 400: this gives 160000.	$\alpha^2 d^2 = 160000.$
Add 160000 to 4680000: this gives 4840000.	$\alpha^2 d^2 + (\alpha^2 + \beta^2)(A - d^2) = 4840000.$
The square root of this is 2200.	$\sqrt{\{\alpha^2 d^2 + (\alpha^2 + \beta^2)(A - d^2)\}} = 2200.$
Add 400 already found: this gives 2600.	$d\alpha + \sqrt{\{\alpha^2 d^2 + (\alpha^2 + \beta^2)(A - d^2)\}} = 2600.$
What part of 5200 gives 2600? Answer: one-half (30 in text).	$\dfrac{d\alpha + \sqrt{\{\alpha^2 d^2 + (\alpha^2 + \beta^2)(A - d^2)\}}}{\alpha^2 + \beta^2}$ $[= \zeta] = \frac{1}{2}.$
½ multiplied by 60 gives 30 as [side of] greater square.	$\frac{1}{2}\beta = \zeta\beta = x = 30.$
Multiply ½ by 40: this gives 20.	$\frac{1}{2}\alpha = \zeta\alpha = 20.$
Subtract 10 from 20 and this gives 10 as [side of] lesser square."	$\zeta\alpha - d = y = 10.$

Surely such work without "any algebraic notation or any sign of a general *theory* underlying the examples is wonderful in itself; it is equally extraordinary that these developments in arithmetic and algebra should have remained, for most of 1800 years at all events, unknown to, or at least without (so far as we can judge) any traceable effect upon, the Greek pioneers in the same subject." [15]

Let us now consider achievements of the Egyptians in pure and applied mathematics. We have referred to the mace record of 3500 B.C. The erection of the great pyramid at Gizeh about 2900 B.C. must have involved many mathematical problems. Here was a huge structure covering more than 13 acres which, with its marvellous connecting roadway to the Nile, which took 100,000 workmen 30 years to build. Over 2,000,000 blocks of stone, averaging 2½ tons in weight, and fitted together with great exactness, were brought from sandstone quarries on the opposite side of

the Nile. For the roofs of chambers granite blocks 27 feet long and four feet thick, weighing 54 tons each, were transported from the quarry more than 600 miles away, and placed in their position over 200 feet above the ground level.

But the problems of mechanics and engineering involved in handling even the larger stone blocks of the great pyramid were slight as compared with those dealt with by the Egyptian in quarrying and setting up some of their huge obelisks of pink granite. The largest existing obelisk, quarried about 1500 B.C. was no less than 105 feet long, nearly 10 feet square at the larger end, and about 430 tons in weight. It was set up in front of the Temple of the Sun at Thebes.

Another extraordinary recorded fact concerning the engineers or surveyors contemporary with the pyramid or obelisk builders is that they were already in possession of methods for locating nilometers around innumerable bends in the river Nile, for a distance of 700 miles, such that the zero points of the nilometers, always below lowest water, are all in one plane.

Most of our knowledge of Egyptian mathematics is derived from two mathematical papyri, the one written about 1850 B.C., usually called the Moscow papyrus,[9] containing 25 problems; and the other dating from about 1650, commonly called the Rhind mathematical papyrus,[7,8] with 85 problems. All of the 110 problems are numerical and many of them are excessively simple. Those in the Rhind papyrus are preceded by a table giving the equivalents in unit fractions of 2 divided by all odd numbers from 5 to 101; for, with the exception of 2/3 the Egyptian had no notation for a fraction with a numerator other than unity. Two divided by 7 is expressed as the sum of 1/4 and 1/28; 2 divided by 97 is expressed as the sum of 1/56, 1/679, and 1/776. Both of these results, as well as 2/3 expressed as the sum of 1/2 and 1/6, are used in a single problem later. The table was therefore a reference list for use in solving problems.

The Egyptian carried through the multiplication of two numbers by successive multiplication of one of the numbers by twos or tens or by 2/3, or by the division of the number by twos or tens. A sort of transposition was also a common operation. For example, if it had been found that 1/5 of 105 was 21 the Egyptian might at once write down 1/21 of 105 is 5. Problems in division are reduced to those of multiplication. It is rather extraordinary that in order to get *one*-third of a number, the Egyptian first found *two*-thirds of the number and then took one-half of the result. This is illustrated in more than a dozen problems of the Rhind papyrus.

Nearly a score of the 110 problems are such as we would now solve by algebra with equations of the first degree in one unknown quantity. For example: "A quantity, its 2/3, its 1/2, and its 1/7,

added together, becomes 33. What is the quantity?" The method of solution used is generally that of trial, or false position, and in more than one case it is obvious that the idea of proportion was clearly understood.

Another score of the problems deal with such questions as the strength of bread and of different kinds of beer, the derivation of beer of great alcoholic strength from two others, and the exchange of beer for bread. An example of this type of problem is the following: "Given that 13 hekat of upper Egyptian grain is made into 18 des of besha date-substitute beer, and that 1 des of this makes $2\frac{1}{6}$ des of barley beer, what is the strength of the barley beer?"

The feed for geese, cranes, ducks, quails, doves, and also for bulls and common cattle is discussed in other problems. The following illustrates a rule-of-three problem: "A sandal maker works for 15 days receiving wages every five days. If he does the work in 10 days after what periods should he be paid?" Of problems in arithmetic progressions the following may be mentioned: "Divide 100 loaves among 5 men in such a way that the share received shall be in arithmetic progression and that one seventh of the sum of the largest three shares shall be equal to the sum of the smallest two." Remember that these are problems of 1650–1850 B.C.

Twenty-six of the 110 problems are geometric and both volumes and areas are discussed. The area of a circle is repeatedly taken as the square of 8/9 of the diameter; this leads us to the remarkable value $256/81 = 3.1605 \ldots$ for π, much better than the somewhat earlier Babylonian value 3. The volume of a right circular cylindric granary is taken as equal to the area of its base multiplied by the number of units in its height. The most recent discussion seems to make it clear that the Egyptian knew that in any triangle its area is equal to one half the product of its base and altitude.[10] The cotangents of the angles with the faces of pyramids make with their bases are discussed. A numerical problem appears to prove the extraordinary fact that the Egyptian knew our formula for the volume of a frustrum of a square pyramid $V = (h/3)(a^2 + ab + b^2)$, where a and b are the lengths of the sides of the square bases and h is the number of units height of the frustrum. The editor of the Moscow papyrus, which was first completely published last year,[9] believed that yet another problem gave the correct result for the area of a hemisphere, if we take the value of π to which reference was just made. But within the past six months a prominent English Egyptologist has concluded that there is no proper ground for this conclusion.[11]

There is no document to prove that the Egyptian knew even a particular case of the Pythagorean theorem.[7]

For 1000 years after 1650 B.C., the approximate date of the Rhind papyrus, there was no new mathematical development of importance, unless it were in connection with the application to such things as sun-dials, water-clocks, or the making of astronomical observations. Hence we conclude our brief consideration of the mathematics of the Babylonians and Egyptians in our first period of 2900 years from 3500 B.C. to 600 B.C. At this time there was no other mathematics than theirs worth considering.

B. Greek Mathematics 600 B.C.–500 A.D.[15]

Greek history began with the third or fourth millenium B.C. but in connection with the history of mathematics, of sculpture, of architecture, of art, of philosophy, of literature, of thought, the semi-millenium commencing about 600 B.C. is of the greatest importance. The Greeks were no longer confined to the Greek peninsula, as in earlier times. They occupied the islands of the Aegean, the western seaboard of Asia Minor, the coasts of Macedonia and Thrace, of Southern Italy and Sicily. Scattered settlements were also to be found as far apart as the mouth of the Rhone, the north of Africa, and the eastern end of the Black Sea. Such was the location of the people who were presently to set such marvellous everlasting beacon lights of Truth, Reason Freedom, Beauty, Excellence, Fellowship between man and man, which were to inspire, guide, and sustain through the milleniums to follow.

What were the special aptitudes which the Greeks possessed for science? Let the greatest living authority on Greek mathematics make answer.[12]

"They had, first, a love of knowledge for its own sake, amounting, as Butcher says, to an instinct and a passion; secondly, a love of truth and a determination to see things as they are; thirdly, a remarkable capacity for accurate observation. Fourthly while eagerly assimilating information from all quarters from Egypt and Babylon in particular, they had an unerring instinct for taking what was worth having and rejecting the rest. As one writer has said, 'it remains their everlasting glory that they discovered and made use of the serious scientific elements in the confused and complex mass of exact observations and superstitious ideas which constitutes the priestly wisdom of the East, and threw all the fantastic rubbish on one side.' Fifthly, they possessed a speculative genius unrivalled in the world's history.

"It was this unique combination of gifts which qualified the Greeks to lead the world in all the intellectual pursuits that make life worth living.

"Last but not less important, the Greeks possessed the advantage over the Egyptians and Babylonians of having no priesthood which could monopolise learning as a preserve of his own, with the inevitable result of sterilising it by keeping it bound up with religious dogmas and prescribed and narrow routine."

Some of these attributes were possibly more in evidence in their great achievements in the fields of medicine, biology, and natural science, than in those of mathematics and astronomy, which now concern us for a few moments. Since hours would be necessary for any adequate description of their wonderful achievements in these fields, we must confine ourselves largely to references to a few names and results.

Greek theoretical geometry and astronomy began with **Thales** of Miletus, on the West coast of Asia Minor, in the first half of the sixth century B.C. No wonder he was declared one of the Seven Wise Men since, apart from being a mathematician and astronomer, he was also a statesman, engineer, man of business, and philosopher. To Thales are attributed the following theorems in elementary geometry:

1. A circle is bisected by any diameter;
2. The angles at the base of an isosceles triangle are equal;
3. If two straight lines cut one another, the vertically opposite angles are respectively equal;
4. If two triangles have two angles and one side in each respectively equal the triangles are equal in all respects.
5. The angle in a semi-circle is a right angle (which we have seen was already recognized by the Babylonians some 1400 years earlier).

Of practical problems he showed how to determine the distance of a ship from the shore and found the height of a pyramid by means of the shadows cast on the ground at the same moment by the pyramid and a stick; that moment was chosen when the length of the stick and its shadow were equal. There is good evidence that Thales predicted the solar eclipse which took place on 28 May 585 B.C. The basis of this prediction was probably the result of observations made by the Babylonians. It may be of interest to remark that Thales was the first known individual with whom definite mathematical discoveries were associated.

About half a century after Thales came **Pythagoras.** With him for the first time geometry was pursued as a study for its own sake. A man of great ability and a most interesting and magnetic mystic, he finally settled at Crotona on the southeastern coast of Italy. Here among the young men of well-to-do families he established a secret society or brotherhood most of whose mathematical discoveries were

pooled. According to Heath,[15] it is to the Pythagoreans that the following geometric results are due:

1. The properties of parallels and their application to prove that the sum of the angles of a triangle is equal to two right angles. From this were deduced the familiar results concerning the sums of (*a*) exterior and (*b*) interior angles of a polygon.
2. The transformation of areas of rectilinear figures, and the sums and differences of such areas, into equivalent areas of different shapes. To this end they invented the powerful method of application of areas, the main constituent of the geometric algebra by which they effected the geometric equivalent of addition, subtraction, division, extraction of the square root, and the complete solution of the general quadratic equation $x^2 \pm px \pm q = 0$, so far as it has real roots.

I can not take the time to explain just what is involved but may remark in passing that the names parabola, ellipse, and hyperbola, were derived from the fact that these curves were respectively defined by the application of an area, the application of an area falling short, the application of an area exceeding; that is $y^2 = px$, $y^2 = px - \frac{p}{d}x^2$, and $y^2 = px + \frac{p}{d}x^2$.

3. The Pythagoreans had a theory of proportion pretty fully developed, though it was only applicable to commensurable magnitudes, being presumably a numerical theory. They were aware of the properties of similar figures such as similar rectilinear figures being in the duplicate ratio of corresponding sides.
4. They had discovered, or were aware of the existence of, at least three of the regular solids—the tetrahedron, cube, and dodecahedron.
5. They discovered the existence of the incommensurable in at least one case, that of the diagonal of a square in relation to its side, and they devised a method of obtaining closer and closer approximations to the value of $\sqrt{2}$ in the form of numerical fractions, which are the successive solutions * of the indeterminate equations $x^2 - 2y^2 = \pm 1$.

Arithmetic in the sense of the theory of numbers began with Pythagoras. With the Greeks Arithmetic, dealing with absolute numbers, or numbers in the abstract, was distinguished from Logistic, the science of calculation, which dealt with numbered things or concrete numbers.

* $y = 2, 5, 12, 29, \cdots; x = 3, 7, 17, 41, \cdots$.

Two numbers are called amicable or friendly if each equals the sum of the aliquot divisors of the others. Pythagoras gave the first pair 220, 284. The second and third pairs were given the seventeenth century by Fermat and Descartes respectively and the next 61 pairs were obtained by Euler in the eighteenth century. Pythagoras discussed also various forms of figured numbers,—triangular, square, pentagonal, etc., the number of dots in a figure corresponding to the number. Thus $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n + 1)$ is any triangular number; $1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2$ is any square number and so on. Such results were known to Pythagoras. To him is also attributed the formula $m^2 + [\frac{1}{2}(m^2 - 1)]^2 = [\frac{1}{2}(m^2 + 1)]^2$, where m was any odd number. But the most wonderful discovery about numbers which he made was in showing the dependence of musical intervals upon numerical ratios. He discovered that for strings of the same tension, the different lengths of strings would be in the ratio of 2 to 1 for the octave, 3 to 2 for the fifth, and 4 to 3 for the fourth. Pythagoras went on to form a diatonic scale and thus initiated a study which was to be extensively elaborated by later Greeks. In the field of astronomy also, Pythagoras was highly original. He was the first to argue that the earth was spherical and that the planets moved in independent circles He associated musical notes with the heavenly bodies (the highest with the fixed stars, the next highest with Saturn, the lowest with the Moon) and also various numbers.

And so, it was with Pythagoras that geometry, arithmetic, music, and astronomy came to be grouped together as fundamental liberal arts for study and to form what was, in the middle ages, called the *quadrivium*. The word "mathematics" is derived from a Greek word *μάθημα* meaning simply a "subject of instruction"; but by the time of Aristotle (−340) the term was definitely restricted to subjects of the *quadrivium*.

The Pythagorean difficulties in connection with geometric proportion and incommensurables were completely solved in the fourth century B.C. by a pupil of Plato named **Eudoxus,** born at Cnidos on the west coast of Asia Minor. He was an original genius second only to Archimedes. His masterly setting forth of incommensurables, as in the fifth book of Euclid's *Elements*, is practically identical with the modern formulation of Dedekind. Eudoxus discovered also the so-called "method of exhaustion" by means of which he gave the first rigorous proof of the results for the volume of a cone, and of a pyramid, and probably showed that: (1) the areas of two circles are to one another as the squares of the lengths of their diameters; and (2) the volumes of spheres are to one another as the cubes of the lengths of their diameters.

It seems certain that **Menaechmus,** a pupil of Eudoxus, was the discoverer of the conic sections,—parabola, ellipse, hyperbola, which were originally thought of as sections perpendicular to generators of right-angled, acute-angled, and obtuse-angled cones. He showed that by means of the intersection of two parabolas, or of a parabola and a rectangular hyperbola so obtained, we could find two mean proportionals between two lines of lengths a and b. This implies the recognition of a geometrical relation, in the case of the parabola, equivalent to our ordinary Cartesian equation, and of a similar relation for a rectangular hyperbola referred to its asymptotes, as axes.

This discovery of Menaechmus was of particular interest because it gave a new solution of the problem of the duplication of the cube, one of the three famous problems which had been formulated in the fifth century B.C. These problems are [18,19]

(*a*) To find a line which shall be the edge of a cube whose volume is double that of a given cube, the problem of the duplication of the cube.
(*b*) To trisect any given angle.
(*c*) To find a line which shall be the side of a square whose area shall be exactly equal to that of a given circle, the problem of squaring the circle.[20]

All of these problems were solved by the Greeks within a century, but more than 22 centuries were to pass before it was finally proved that no one of them could be solved with ruler and compasses alone. Already in the fifth century B.C., by means of a curve called the quadratrix,[21] the problem of the trisection of an angle was solved by **Hippias** of Elis, and **Hippocrates,** of Chios, had shown that the problem of duplicating the cube could be reduced to that of finding two mean proportionals between such lengths as a and $2a$. Such mean proportionals were found by **Archytas,** a Pythagorean, a friend of Plato, and a teacher of Menaechmus, in a marvellous construction by means of the intersection of a right cone, a cylinder, and an anchor ring with inner diameter zero. **Dinostratus,** a brother of Menaechmus, showed that the quadratrix of Hippias could also be used to solve the problem of squaring the circle. In the third century this problem was also solved by a spiral invented by Archimedes.[21] In the second century B.C. the quartic curve called the conchoid of **Nicomedes** [21] was used to solve the problem of trisecting an angle, and the cissoid of **Diocles,**[21] a cubic curve, was employed for solving the problem of the duplication of the cube.

We come now to the consideration of the golden period of Greek mathematics and of the greatest mathematical school of ancient times. This was at the magnificently endowed university of

Alexandria which had been founded by Alexander the Great in 332 B.C. The university was opened about 300 B.C. and within the first 40 years of its existence over 600,000 rolls had been collected in its great library. **Euclid** was the university's first professor of mathematics. Practically nothing is known about his life except that he was the author of at least ten treatises of which approximately complete texts of five are available. These include three on applied mathematics, namely on spherical astronomy, on optics, and on music. But by far the most famous one is his treatise, in 13 books, called the *Elements*.[22] More than a thousand editions have appeared since the first one printed in 1482, and for 1800 years before that manuscript copies dominated all teaching of geometry.[4] Though a large portion of the subject matter had been investigated by predecessors the whole arrangement was due to the great genius of Euclid who supplied innumerable details. It is quite impossible to give in a few sentences any adequate idea of the contents of the 465 propositions in this monumental work. Practically all of the geometrical material of American school texts in plane and solid geometry is contained in parts of six of the books (1, 3, 4, 6, 11 and 12) of the *Elements*. Most books are prefaced by definitions, but before the first book are certain postulates the fifth of which is the famous one which differentiates Euclidean from non-Euclidean geometry.[41] The statement is as follows: "Let it be postulated that, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines if produced indefinitely, meet on that side on which are the angles less than two right angles."

In the Book II of the Elements are a number of propositions giving geometric proofs of algebraic identities, such as $(a + b)^2 = a^2 + 2ab + b^2$, and in Books II and VI are the propositions giving, among many other things, the solutions of quadratic equations, which we have already noted as discoveries of the Pythagoreans. In Book V is expounded the remarkable Eudoxus theory of proportion, alike applicable to incommensurable and commensurable magnitudes of every kind. There are many masterly Euclidean developments. Book VI applies the general theory of properties set out in Book V to plane geometry. The 102 propositions in Books VII, VIII, IX deal with questions in the theory of numbers. Special reference may be made to three propositions in Book IX. It is elegantly proved in the 20th that there are an infinite number of prime numbers; in the 35th there is a beautiful geometrical derivation of what is practically equivalent to our algebraic formula for the sum of the first n terms of a geometric progression; and in the 36th it is proved that if $S_n = 2^n - 1$ is prime, then $P = 2^{n-1}S_n$ is a perfect number,[23] that is, a number

which is equal to the sum of its divisors smaller than itself. The first four perfect numbers are 6, 28, 496, and 8128; only 12 perfect numbers are known, and they are all of the Euclidean type.

Book X contains 115 propositions on incommensurables, and is perhaps the most remarkable and most finished of all the books of the elements. Books XI, XII, XIII deal with geometry of three dimensions. In propositions 16 and 17 of the last book all the details are carried out for the actual construction of an icosahedron, and of a dodecahedron, inscribed in a sphere.

The logical form of the presentation of the propositions in the *Elements* is especially notable. There is first of all the enunciation of the proposition; secondly, the statement of the precise data; thirdly, the statement of what we are required to do with reference to the precise data mentioned; fourthly, the construction, the addition when necessary of more lines to the figure; fifthly, the proof; and sixthly, the conclusion, stating what has actually been done, which generally follows the wording of the original enunciation.

In the period between Euclid and Archimedes came **Aristarchus** of Samos,[24] whose great achievement lies in the fact that he was the first to place the sun in the center of the universe, asserting that the earth and the other planets (Venus, Mercury, Mars, Jupiter, and Saturn) revolved about it, thus anticipating Copernicus by seventeen centuries.

Archimedes [25] was born at Syracuse, in Sicily, about 287 B.C. and was killed in the sack of that city by the Romans in 212. He studied with successors of Euclid at Alexandria and it was probably in Egypt that he invented the water screw, known by his name, for drawing water to irrigate fields. On returning to Syracuse he devoted his time wholly to mathematical research. Up to the time of Newton at least he was the greatest mathematical genius that the world had seen. The following summary of his work is mainly due to Heath.[15]

In geometry his works consisted chiefly of original researches in the quadrature of plane curvilinear figures and in the quadrature and cubature of curved surfaces. By methods equivalent to integration Archimedes found the area of a parabolic segment, the area of a spiral, the surface and volume of a sphere and a segment of a sphere, and the volumes of any segments of the solids of revolution of the second degree.

In arithmetic he calculated approximations to the value of π and in the course of these calculations showed that he could find approximate values for the square roots of large or small non-square numbers; he found $22/7 > \pi > 3\ 10/71$, and also, apparently, $3.141697 \cdots > \pi > 3.141495 \cdots$, the mean of which is 3.141596. Archimedes also invented a system of arithmetic nomenclature by

which he could express in language enormously large numbers in fact all numbers up to that which we would write as 1 followed by 80,000 million million ciphers.

In mechanics he laid down certain postulates and, on the basis of these postulates established certain fundamental theorems on magnitudes balancing about a point and on centers of gravity, going so far as to find the center of gravity of any segment of a parabola, a semi-circle, a cone, a hemisphere, a segment of a sphere, and a right segment of a paraboloid of revolution. As we learn from one of his works, discovered as recently as 1906, Archimedes made most ingenious use of mechanics as an aid for suggesting probable results in geometry; the first theorem which he found in this way was in connection with the quadrature of the parabola.

Lastly, he invented the whole science of hydrostatics, which again he carried so far as to give a complete investigation of the positions of rest and stability of a right segment of a paraboloid of revolution floating in a fluid with the base either upwards or downwards, but so that the base is entirely above or entirely below the surface of the fluid. Heath then sums up as follows:

"The treatises themselves are, without exception, models of mathematical exposition; the gradual unfolding of the plan of attack, the masterly ordering of the propositions, the stern elimination of everything not immediately relevant, the perfect finish of the whole, combine to produce a deep impression, almost a feeling of awe in the mind of the reader."

In accordance with the desire of Archimedes the figure of a sphere inscribed in a cylinder was engraved on his tomb since he seemed to regard the discovery that the volume of the sphere was two thirds that of the circumscribed cylinder as his greatest achievement. When Cicero was queaestor in Sicily in 75 B.C., he found the tomb with this inscription.

In conclusion we may note two results due to Archimedes: (*a*) our well known formula for the area of a triangle in terms of its three sides; (*b*) the solution of a three-term cubic equation, the term of the first degree being lacking, by means of the intersection of a parabola and a rectangular hyperbola; this occurs in his work on the sphere and cylinder.

Almost contemporary with Archimedes was his friend **Eratosthenes,** who taught in Alexandria and was librarian at the University. To him Archimedes dedicated his treatise on *Method.* The most famous scientific achievement of Eratosthenes was his determination of the polar circumference of the earth correct to within about 50 miles. This was done by observing that when the sun was in the zenith at Assouan, it was exactly 7° 12′ south of the zenith at Alexandria, known to be 5000 stadia distant. According to a recent

interpretation,[27] an Eratosthenes' device, called a sieve, for determining prime numbers, is of considerable interest.*

The third great name in connection with the Alexandrian School was **Apollonius** of Perga who was about 25 years younger than Archimedes. He was called by his contemporaries the 'Great Geometer,' because of his extraordinary treatise on *Conics*, and he was also mentioned as a famous astronomer. He studied at Alexandria. Only seven of the eight books of his *Conics* have been preserved. The first three seem to correspond to a treatise which Euclid wrote, but books V, VI, VII contain the discoveries which Apollonius himself had made. To him are due the terms parabola, ellipse, hyperbola, for reasons which we have already explained, and his lines of reference are any diameter and a tangent at its extremity, the cases of these being at right angles being considered only as a special cases. Book V treats of various questions concerning normals to conics and from certain propositions we can readily deduce the Cartesian equations to the evolutes of the three conics. Book VII deals mainly with various results in connection with conjugate diameters. There are nearly 400 propositions. The suggestions that I have given are sufficient to indicate that in this work of Apollonius there is far more than is contained in any of our American textbooks on analytical geometry, so far as conic sections are concerned.

Among a number of other works of Apollonius we shall refer to only two. In his *Plane Loci* the following familiar results were given: (*a*) If A, B be fixed points and AP and BP are in a given ratio, the locus of P is a straight line or a circle according as the given ratio is or is not one of equality; (*b*) If A, B, C, $\cdots$ be any number of fixed points and α, β, γ, $\cdots$ any constants, the locus of point P, such that $\alpha \cdot AP^2 + \beta \cdot BP^2 + \gamma \cdot CP^2 + \cdots =$ a constant, is a circle.

The second work, on *Contacts* or *Tangencies*, has not come down to us, but its principal problem is to describe a circle tangent to three given circles, which is usually known as the problem of Apollonius. Many celebrated mathematicians, such as Viète and Newton, have worked on this problem, and what is known of the work as a whole has made it the basis of numerous restorations.

We have now already seen the marvellous development of geometry by the Greeks from the beginning up to difficult problems of the integral calculus, and all of this within the short space of 350 years.

* In E. Hoppe's *Mathematik und Astronomie im klassischen Altertrum* (Heidelberg, 1911), it is stated on page 284 that the method of the sieve of Eratosthenes for finding prime numbers was set forth completely in section 52 of Plato's *Phaedo*. This statement has been more than once recently quoted. On consulting the section in question, I find not the slightest basis for the statement's substantiation.

The next name of mathematical interest among the Greeks is that of **Hipparchus,** probably the most eminent astronomer of antiquity, who flourished about 140 B.C. His greatest discovery was the precession of the equinoxes.[28] But the science of trigonometry seems to have begun with him, and also the division of the circle in general into 360°. The former is inferred more particularly from the early part of the *Almagest* of **Claudius Ptolemy** of Alexandria who flourished in the second century of our era. He gave a table of chords which is equivalent to the sines of angles ¼°, ½°, ¾°, and so on by quarters of a degree to 90°.

If in a circle of unit diameter a chord subtends at the center an angle 2α it would subtend an angle α at the circumference and hence the chord would equal to $\sin \alpha$. Hence it was easily observed that $\sin^2 \alpha + \cos^2 \alpha = 1$. Ptolemy's theorem regarding a quadrilateral inscribed in a circle is given, and from this, the well-known formulæ for $\sin (A \pm B)$ and $\cos (A \pm B)$, among others, are readily derived.

Ptolemy used the stereographic projection in making a map. The essentials of his argument in an attempted proof of Euclid's parallel postulate have been preserved. It is recorded that he wrote also a work on dimension in which he attempted to prove that the possible number of dimensions is three only.

We next come to **Heron** of Alexandria who probably flourished after Ptolemy, although his date has been a matter of uncertainty, even to the extent of 400 years. He was an almost encyclopedic writer on mathematical and physical subjects, and aimed at practical utility rather than theoretical completeness. In his *Pneumatica* are many mechanical devices such as a siphon, a fire engine, a device whereby temple doors are opened by fire on an altar, an altar organ blown by the agency of hand labor or by a windmill, and a jet of steam supporting a sphere. He gave an elegant geometric proof of the formula for the area of a triangle in terms of the sides, now known to be due to Archimedes. His formula for the volume of the frustrum of a square pyramid can be readily reduced to the one used 2000 years earlier in the Moscow papyrus. Quadratic equations are solved in a manner very similar to that of the Babylonians 2200 years before. Incidentally Heron shows how he obtained the cube root of a non-cube number, 100. He found also the volume of an anchor ring and of the five regular solids.

About a century after Heron flourished **Diophantus** [72] of Alexandria, one of the greatest mathematicians of the Greek civilization. He was the first to make systematic use of symbols in algebraic work, a sign for unknown, a sign for minus, and signs for the various powers,—square, cube, etc. Of his great work called *Arithmetic* only six of its thirteen books survive and they are mostly taken up with

problems in indeterminate analysis of the second degree. Hence the term Diophantine Analysis. The answers are always in positive rational numbers. The collection is extraordinarily varied and the devices resorted to are highly ingenious. The following are samples of the problems:

(*a*) Given a number, to find three others such that the sum of the three, or of any pair of them, together with the given number, is a square.

(*b*) To find three numbers such that the product of any two, plus or minus the sum of the three, is a square.

And finally we come to **Pappus** of Alexandria who lived at the end of the third century. His great work, entitled *Mathematical Collections*, covers the whole range of Greek geometry and was intended "to be read with the original works (where extant) rather than independently. Where, however, the history of a subject is given, e.g., that of the duplication of the cube, or the finding of the two mean proportionals, the solutions themselves are reproduced, presumably because they were not readily accessible but had to be collected from scattered sources. Even when some accessible classic is being described, the opportunity is taken to give alternative methods or to make improvements in proofs, extensions and so on. Without pretending to great originality, the whole work shows, on the part of the author, a complete grasp over the subjects treated, independence of judgment, and mastery of technique; the style is concise and clear; in short Pappus stands out as an accomplished and versatile mathematician and a worthy representative of the classical Greek geometry." [15] Among the original contributions of Pappus are,—(*a*) a generalization of the Pythagorean theorem to any triangle with certain parallelograms on its sides; and (*b*) a result often attributed to Guldin, a Dutch mathematician of the seventeenth century, The volume of a solid of revolution is equal to the area of the generating figure, multiplied by the circumference described by the center of gravity of the figure. The fifth book is devoted mainly to the subject of isoperimetry. There is a very interesting passage concerning bees, their orderliness, and the hexagonal form of their cells exactly filling up space about a point. It is later shown that (*a*) a circle is greater than any regular polygon of equal contour; (*b*) a sphere is greater than any of the five regular solids with equal area.

Alexandria was razed by the Arabs in the seventh century. But they destroyed the library, in the public bath of the city in 389. It is said that it took six months to burn all the rolls.

C. Hindu, Arabic, Persian Mathematics, 500–1200[30]

With Pappus creative Greek mathematicians came to an end, and very soon all the great traditions of Greek learning had died out; then followed nearly a thousand years in which only comparatively slight additions to the sum of mathematical knowledge were made. Some of these slight contributions came from India.

The formula for the area of an inscribed quadrilateral, similar in form to that for a triangle, ascribed to Archimedes, was first given by **Brahmagupta,** in the early part of the seventh century, but it was not recognized as true only for a cyclic figure. If a, b, c, and d are the lengths of the sides of the quadrilateral, $s = \frac{1}{2}(a + b + c + d)$, and m and n are the lengths of the diagonals, the area is equal to $\sqrt{[(s-a)(s-b)(s-c)(s-d)]}$, $m = \sqrt{[(ab+cd)(ac+bd)/(ad+bc)]}$, and $n = \sqrt{[(ac + bd)(ad + bc)/(ab + cd)]}$. These formulas were considered with a view to determining quadrilaterals whose sides, diagonals, and areas were all rational quantities. In particular Brahmagupta gave the rule: If $a^2 + b^2 = c^2$, and $\gamma^2 + \beta^2 = \gamma^2$, then the quadrilateral $(a\gamma, c\beta, b\gamma, ca)$ is cyclic, is rational, and has its diagonals at right angles.

We have already seen that the Pythagoreans were led to solutions of the equations $x^2 - 2y^2 = \pm 1$, in getting approximations to $\sqrt{2}$. Brahmagupta and **Bhāskara** (of the twelfth century) gave the remarkable, complete, solutions for the general equations $x^2 - Ay^2 = \pm 1$, A being an integer. These are the most important developments in Hindu mathematics. Brahmagupta gave also the general solution of $ax + by = c$, as $x = cq - bt$, $y = -cp + at$, where t is zero or any integer, and p/q is the second last convergent of a/b, expressed as a continued fraction.

It may be appropriate at this point to make some remarks concerning the origin of our common numerals.[31] So far as we know the largest number of our numeral forms were first used in India.[32] The 1, 4 and 6 are found in inscriptions of the third century B.C.; the 2, 4, 6, 7, and 9 appear in another inscription about a century later; and the 2, 3, 4, 5, 6, 7, and 9 in caves of the first and second century of our era, all in forms which have a considerable resemblance to our own.

The first definite external reference to Hindu numerals is in a note by a Bishop who lived in Mesopotania about 650 A.D. Early in the ninth century the numerals became known to Arabic scholars. Indeed, one of the most prominent of these, **Mohammed ibn Mûsâ al-Khowârizmî** of Baghdad, wrote a treatise on the subject, and this was translated into Latin in the twelfth century. The earliest undoubted occurence of zero in India was in an inscription of 876, in connection with the numbers 50 and 270. But it was

used much earlier, indeed at the beginning of the Christian era, in astronomical records of the Mayas of Central America.[33] They had a vigesimal system with principle of local value. About two centuries before Christ the Babylonians possessed the principle of local value in calculations and a zero symbol which was not, however, used systematically in writing numbers, and not at all in computations. In Europe the complete system of numerals with the zero was derived from the Arabs in the twelfth century.

We have referred to the astronomer and mathematician al-Khowârizmi. He wrote a work on algebra with the title: "'ilm al-jabr wa'l muqabalah," which has been translated "the science of reduction and cancellation." The Arabic word for reduction, "al-jabr," thus became our word algebra. It should be remarked, however, that a recent writer [34] has argued that the correct translation of the above title is "science of equations" the word "al-jabr" being an Assyrian word for equations, and "al-muqabalah" the Arabic word for equations.

From the time of the Pythagoreans the constructions of elementary geometry were imagined as carried through with a ruler and with compasses with variable opening. A prominent Arabian mathematician of the tenth century, **Abû'l Wêfa,** gave some lectures, preserved by a student, on geometrical constructions with a ruler and compasses with a fixed opening. It was not till centuries later that it was recognized that there was no real limitation here, since it was shown by Poncelet and later by Steiner, in the early part of the nineteenth century, that if a single circle and its center are once drawn in a plane, every construction with ruler and compasses can be carried through with ruler alone.

The ninth and tenth centuries may be regarded as the golden age of Arabian mathematicians to whom the world owes a great debt for preserving and transmitting to posterity many classics of Greek mathematics, for their contribution was mainly that of transmission.

In concluding the period under consideration we should make brief reference to work of **Omar Khayyàm,**[35] the tent-maker,[36] known to the western world as the author of the Rubaiyat. He wrote a treatise on algebra in which his geometrical treatment of the cubic equation is central. He obtains a root as the abscissa of a point of intersection of a conic and a circle, or of two conics. Various forms of cubic equations are considered; negative roots are rejected, and not all positive roots are discovered. While we have already noted the solution of cubic equations by Menaechmus and Archimedes it is to be remarked that here the point of view is different; the problem is: How can we solve cubic equations with numerical coefficients? Another notable achievement of Omar was his correction of the calendar by the introduction of cycles of 33

years. This calendar was more accurate than the one we use to-day. Omar was the greatest and most original of Saracen mathematicians.

D. European Mathematics 1200–1600

During the period 500–1200 the student went to the teacher in the monastery and heard his lectures. But in the thirteenth century universities commenced to spring up at such places as Bologna, Padua, Naples, Paris, Oxford, and Cambridge. Scribes making copies of treatises were thus kept busily employed by the universities. By the middle of the fifteenth century, however, their products were being sold as books are today. But such methods of disseminating knowledge were crude when compared with that of the distribution of the printed work. The publication of these with movable type commenced about 1450. More than two hundred mathematical works were printed, in Italy alone, before 1500; but this number was increased to 1527 in the next century.

During three and a half of the four centuries now under consideration Italy made the chief contributions to mathematics, and by far the most outstanding mathematician was one who flourished at the beginning of this period, **Leonardo** of Pisa, often called Fibonacci, that is, son of Bonaccio. During early life he travelled extensively about the Mediterranean visiting Egypt, Syria, Greece, Sicily, and southern France, and knowledge thus gleaned regarding arithmetic systems used by merchants of different countries was the basis of a notable work, entitled *Liber Abaci*, which he wrote in 1202. This is a storehouse from which for centuries authors got material for works on arithmetic and algebra. The Hindu-Arabic system of numerals was here strongly advocated and illustrated, and the work did much to introduce it into Europe. Leonardo recognized that a quadratic equation might have two roots but did not recognize those negative or imaginary. It is in this work that we find the problem: "7 old women invaded Rome; each woman had 7 mules; each mule carried 7 sacks; each sack contained 7 loaves; and with each loaf were 7 knives; each knife was put up in 7 sheaths. What is the sum total of all named?" It was this problem which gave the clue to the interpretation of a problem in the Rhind papyrus 2800 years earlier, "In each of 7 houses are 7 cats, each cat kills 7 mice, each mouse would have eaten 7 ears of spelt, and each ear of spelt will produce 7 hekat of grain; how much grain is thereby saved?" The modern conundrum starting out: "As I was going up to St. Ives I met a man with seven wives, each wife had seven sacks," etc., further illustrates the perpetuation of this type of problem through the centuries. An interesting chapter on similar perpetuation of other problems has been written.[37]

In *Liber Abaci* is also the following: How many pairs of rabbits can be produced from a single pair in a year if it is supposed: (*a*) that every month each pair begets a new pair which from the second month on becomes productive; and (*b*) that deaths do not occur? In this way we are led to the famous recurrent Fibonacci series: [38] 1, 1, 2, 3, 5, 8, 13, 21, $\cdots$, in which each term is the sum of the two preceding.

Among many other things the work contains also a proof of the well-known algebraic identity expressing the product of the sum of two squares as the sum of two squares: $(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (bc - ad)^2 = (ad + bc)^2 + (bd - ac)^2$.

Leonardo wrote two other important works: *Liber Quadratorum* [39] in 1220 and *Practica Geometriae* in 1225. The first of these is a brilliantly written, original, and able work on indeterminate analysis, stamping the author as the outstanding mathematician in the field from the time of Diophantos to the time of Fermat over 400 years later. The great *Practica Geometriae* brings together a vast amount of material in geometry and trigonometry, and it would seem as if some works of the ancients now lost had been still available to Leonardo. In particular this seems to have been true of Euclid's work on the *Division of Figures*.

Leonardo's great reputation led to a sort of mathematical tournament being held when problems previously given to the contestants were solved. That Leonardo was able at this time to find correctly to 9 places of decimals a root of a four-term cubic equation with numerical coefficients, has excited great wonder. There is no clue as to what his method of arriving at the result may have been.

Skipping over a period of about 250 barren years we come to a German named **Johann Müller,** born near Königsberg, and as a result known in the history of mathematics as **Regiomontanus.**[41] He was the most influential and best-known German mathematician of the fifteenth century. We shall simply note concerning him that his work on trigonometry, written about 1464, but first published in 1533, was the first modern systematic exposition of plane and spherical trigonometry; the only functions introduced are sines and cosines. The work had great influence in establishing trigonometry as a science independent of astronomy. The period of Regiomontanus is also that in which two mathematical works were printed, namely the first dated arithmetic [40] (Treviso, 1478), an anonymous commercial work, and the first edition of Euclid's *Elements*, a Latin translation by Campanus (Venice, Ratdolt, 1482).

In the sixteenth century the chief Italian achievement was the solution of equations of the third and fourth degree. The facts with regard to the general solution of the cubic equation seem to be as follows:

Scipio del Ferro, a professor of mathematics at the University of Bologna, solved the equation $x^3 + mx = n$ in 1515, imparting the result to his pupil **Antonio Fior,** without publication. About 1535 **Nicolo** of Brescia who stammered badly because of an injury received as a child, and was therefore called **Tartaglia,** the stammerer, discovered the solution of the cubic $x^3 + px^2 = n$ as well as that of Ferro's form. In a public contest provoked by Fior who had been suspicious of Tartaglia's achievements, Tartaglia triumphed completely. Under a pledge of secrecy Tartaglia confided his method of solution to **Girolamo Cardano,** an unprincipled genius who taught mathematics and practised medicine in Milan. By far the most notable of the numerous works which Cardano wrote was his *Ars Magna* published in 1545. This was the first great Latin treatise devoted solely to algebra. It was here that Tartaglia's general solution of the cubic equation was first published [41] without Tartaglia's consent. Here too appeared the first solution of the general biquadratic equation; [41] this had been found by **Ferrari,** a pupil of Cardano.

Already at this time the astronomers were feeling the need of trigonometric tables. After twelve years of incessant labor with computers, two works of extraordinary merit, and of the greatest value even to the present day, were finally prepared by **Georg Rhaeticus,** but not published till after his death. One of the works was a ten-place table of the natural sines, cosines, and secants, for every ten seconds of arc; the other was a fifteen-place table of natural sines, to every ten seconds of arc, with first, second, and third differences. Rhaeticus was the leading mathematical astronomer in Teutonic countries in the sixteenth century.

The sixteenth century was also that of the Polish astronomer **Copernicus,**[46] who after thirty-six years of labor, published in 1543 his great work *The Revolution of the Heavenly Bodies.* The Copernican Theory thus formulated had a great effect on thought of the time. We have seen that it had been put forward by Aristarchus about 1800 years earlier.

And finally we come to the greatest of all French mathematicians of the sixteenth century, **François Viète,** a lawyer and member of Parliament who devoted most of his leisure to mathematics. His collected mathematical works form a considerable volume. He contributed extensively to the development of algebra and trigonometry. "He was among the first" to use Smith's summary,[4] "to employ letters to represent numbers in algebra, often using vowels for the unknowns, and consonants for the knowns. He found the formula for sin $n\phi$ in terms of sin ϕ; made advances towards proving that an equation of the nth degree is made up of n linear factors; showed how to increase, decrease, multiply, or divide

the roots of the equation $f(x) = 0$, by k; gave one of the earliest methods of evaluating π by infinite products; applied algebra to geometry in such a way as to lay a foundation for analytic trigonometry; indicated powers more simply than his predecessors had done, using Aq for the square of the unknown, Ac for its cube, Aqq for its fourth power, and so on; and showed clearly the relation between the problems of the trisection of an angle and the solution of a cubic equation."

In concluding our notes on mathematical developments during the 5000 years ending with the sixteenth century, we may draw attention to the fact that the first work on mathematics printed in the New World appeared at Mexico City in 1556, within 64 years of the discovery of America. It was the *Summario Compendioso* by **Juan Diez,**[42] and contained (*a*) tables intended to assist merchants in buying gold and silver, (*b*) arithmetic suited to needs of apprentices in counting houses, and (*c*) a few pages of algebraic problems, chiefly relating to quadratic equations.

HISTORY OF MATHEMATICS AFTER THE SIXTEENTH CENTURY

A. The Seventeenth Century

The seventeenth century is especially outstanding in the history of mathematics. It saw Fermat lay foundations for modern number theory, Descartes invent analytic geometry, Pascal and Desargues open up new fields for pure geometry,[4] Kepler discover laws of heavenly bodies, Galileo Galilei reveal laws of mechanics, Huygens make notable contributions in the theory of probability and other fields, Newton create new worlds with calculus, curves, and physical observations, Leibniz build master notations, and Napier double the astronomer's life through calculating devices. One can not help being struck by the fact that in this century, with a single exception, all of these creators in mathematics were to be found in the north where supremacy has reigned ever since, in practically all fields.

While Napier, Kepler, and Galileo each lived many years in the sixteenth century, practically all of their important results to which we shall refer were obtained or announced in the seventeenth. It may be well to consider first the contributions of these men.

John Napier [43] was a Scot, spent most of his life at Merchiston Castle near Edinburgh, and took an active part in the political and religious controversies of the day. As Ball expresses it:[1] "The business of his life was to show that the Pope was Antichrist, but his

favourite amusement was the study of mathematics and science." In connection with the history of mathematics he is usually thought of only as the great inventor of logarithms; [44] but the "rule of circular parts" (a mnemonic for readily reproducing the formulas used in solving right spherical triangles), the four formulas (known as "Napier's analogies") for solving general spherical triangles, and the calculating rods,[41] called "Napier's Rods," for multiplying dividing and taking of square roots, were also products of his genius. Napier seems to have had the idea of logarithms in mind as early as 1594, but it was not till 1614 that he published his researches in a "description of the admirable canon of logarithms." [45] Besides explaining his logarithms, he gives a table of the logarithms of natural sines from 0° to 90° for each minute. The base of these logarithms is sometimes said to be $1/e$; the exact relation is

$$\text{Nap. log } x = 10^7 \log_{1/e}(x/10^7).$$

Napier defined a logarithm by means of relations between moving points on two lines,[41] one of fixed length 10^7, and the other of indefinite extent. While, from the time of Archimedes on to the time of Napier, there were numerous instances of the recognition of such a relation as $a^m \cdot a^n = a^{m+n}$, the law of exponents was in no sense a common idea, and it was not till later that the notion of a logarithm as an exponent became general. In Napier's system the logarithm of 10^7 was zero and logarithm of unity was not zero, although he recognized that modifications of his system to this was desirable. It was effected by his friend, the English mathematician, **Henry Briggs,** whose table of the logarithms of numbers from 1 to 20,000, and from 90,000 to 100,000, to 14 places,* were published in 1624, seven years after Napier's death. Parts of a great new table to 20 places of decimals are now appearing in England in partial celebration of the tercentenary of the discovery of logarithms. In Italy Napier's wonderful discovery was taken up enthusiastically by **Bonaventura Cavalieri,** a pupil of Galileo and professor of mathematics at the University of Bologna for the last 18 years of his life; we shall presently have something to say about his work. In Germany Kepler's zeal and reputation soon brought them into vogue there.[26]

Let us now consider some of the contributions made by the second man of our group, **Johann Kepler,**[46] a prodigious worker, a competent mathematician, and one of the founders of modern astronomy. In 1601 he became an assistant of the remarkable Danish astronomical observer **Tyge Brahe,** who died a little later.

* The gap from 20000 to 90000, to ten places of decimals, was filled in by Adriaen Vlacq in a work published in 1628.

The vast body of accurate observations of the planets which Kepler thus inherited led him in 1609 and 1619 to formulate the following laws:

I. All the planets move round the sun in elliptic orbits with the sun at one focus;

II. The line joining the planet to the sun sweeps out equal areas in equal intervals of time.

III. For all planets the square of the time of one complete revolution (or year) is proportional to the cube of the mean distance from the sun.

Purely as a thrilling intellectual experience, without any imaginable practical application, Apollonius of Perga, and other Greeks, developed a marvellous body of knowledge with regard to conic sections. Then suddenly, 1800 years later, to a Kepler this knowledge had most illuminating practical applications.

Kepler was also a geometrician of some insight. He contributed original results to the theory of star polygons and star polyhedra. In one of his works,[1] which consists rather in the enunciation of certain general principles, illustrated by a few cases, than in a systematic exposition, he laid down what has been called a Principle of Continuity, and gives as an example the statement that a parabola is at once the limiting case of an ellipse and of a hyperbola. He illustrated the same doctrine by reference to foci of conics; he explained also that parallel lines should be regarded as meeting at infinity.

The last member of our group, **Galileo Galilei,**[46] one of the most interesting figures in the history of science, was the founder of the science of dynamics. His work in astronomy we shall not consider. A native of Pisa he carried on experiments from the leaning tower there and arrived at the law that the distance of descent of a falling body was proportional to the square of the time, which is in accordance with the well known law that $s = \frac{1}{2}gt^2$. Galileo was the first to realize that, neglecting air resistance, the path of a projectile is a parabola. He had a clear notion of acceleration, and gave a correct definition of momentum. He suggested that arches of bridges should be built in the form of cycloids, and by weighing pieces of paper surmised that the area of a cycloid is three times that of its generating circle.

Another mathematician who lived many years in the sixteenth century, although his outstanding publication appeared in the seventeenth century, was the Englishman **Thomas Harriot,**[47] who is of special interest to Americans since he was sent in 1585 to survey and map Virginia, now North Carolina. He died in 1621; one of his works published in 1631, *Artis Analyticae Praxis*, includes: the formation of equations with given roots, a statement of relationships

between roots and coefficients, and our symbols for "greater than" and "less than." But in unpublished manuscripts there is a table of binomial coefficients worked out in the form of a Pascal triangle.[41] It was more than thirty years after Harriot's death that Pascal is known to have used this triangle. Morley stated[47] that the manuscripts contained a well-formed analytical geometry but D. E. Smith found that the manuscript in question was not in Harriot's handwriting.

Harriot's interesting posthumous work appeared in the same year, 1631, as the first edition of the *Clavis Mathematicae of* **William Oughtred,**[48] a work on arithmetic and algebra. After Napier's *Descriptio* in 1614, this was the most influential mathematical publication in Great Britain, in the first half of the seventeenth century. It was one of the few books that contributed to laying the foundations of the mathematical knowledge of Newton as he was starting on his career. Oughtred exceptionally emphasized the use of mathematical symbols and although he introduced more than a hundred of them, only three have come down to modern times. These are: our cross-sign for multiplication, the four-dot sign in proportion (: :), and our sign for difference between ($\sim$). About 1622 Oughtred invented the slide rule[49] and his priority in the invention is unassailable. Oughtred was by profession a minister of the gospel but his avocation was the teaching of mathematics and the writing of mathematical books. Among his pupils were John Wallis, and Christopher, afterward Sir Christopher, Wren. He was born in the last third of the sixteenth century and lived to be nearly ninety.

We may appropriately consider next the contributions of a member of the French school, **Gérard Desargues,** born in the last decade of the sixteenth century. By profession an engineer and architect his gratuitous lectures in Paris, when he was in his thirties, made a great impression on his contemporaries. Among his books was a treatise on how to teach children to sing well. But it was his remarkable treatise on conics which stamps him as the most original contributor to pure geometry in the seventeenth century. Desargues commences with a statement of the doctrine of continuity as laid down by Kepler and develops fundamental theorems on involution, homology, poles and polars, and perspective,[41] such as arise to-day in our courses on synthetic projective geometry. This subject was not developed further, to any extent, till the early part of the nineteenth century.

When a mathematician hears the name Pascal he usually thinks of another French mathematician, **Blaise Pascal,** author of the famous *Provincial Letters*, and not of his father Étienne whose name is connected with the quartic curve called Pascal's limaçon. Blaise

showed phenomenal ability in mathematics at an early age and Descartes could not at first credit that the manuscript on conics [50] which he wrote at the age of 16, had been written by him and not by his father. It was here that the famous Pascal's theorem occurs, If a hexagon be inscribed in a conic the points of intersection of opposite sides will be collinear. In 1642, at the age of 19, he invented a computing machine which he later improved.[41] This, and one constructed by Leibniz about 1694, were the first of their kind. Pascal's last work was on the cycloid. In correspondence with Fermat he laid down the principles of the theory of probability.[41,66]

In connection with pure geometry in the seventeenth century it should now be noted that a work published at Amsterdam in 1672 by a Dane named **Georg Mohr** [51] contained an elegant proof that constructions with ruler and compasses could be carried through with compasses alone. Up to the republication of this work, three years ago, by the Danish Society of Sciences, it was generally thought that this theorem was first proved 125 years later by a capable Italian mathematician **Lorenzo Mascheroni,** and the term "Mascheroni constructions" (which should much more correctly be "Mohr constructions") is current in mathematical literature.

The invention of analytic geometry and of the infinitesimal calculus diverted attention from pure geometry. Let us now consider the work of the Frenchman, **René Descartes,**[46] born in a family of wealth and culture, who greatly enriched the world by his writings in many fields. But we are chiefly interested in his *La Géométrie*, an appendix to a large work published at Amsterdam in 1637. The work is obscure and it is by no means a systematic exposition of the new method. The fundamental idea in the mind of Descartes [52] was, not the revolutionizing of geometry, so much as the elucidation of algebra, by means of geometric intuition and concepts; in a word, the graphic treatment of equations. Analytic geometry does not consist in the application of algebra to geometry; that had been done by Archimedes and many others and had become the usual method of procedure in the works of mathematicians of the sixteenth century. The great advance made by Descartes was that he saw that a point in a plane could be completely determined if its distance, say x and y, from two fixed lines in the plane were given, with familiar conventions as to the interpretation of positive and negative values; and that the equation $f(x, y) = 0$ was indeterminate and could be satisfied by an infinite number of values of x and y, yet these values of x and y determined the coördinates of a number of points which form a curve, of which the equation $f(x, y) = 0$ expresses some geometrical property, that is, a property true of the curve at every point on it (as in the definition of a conic).

It was at once seen that in order to investigate the properties of a

curve it was sufficient to select, as a definition, any characteristic geometrical property and to express it by means of an equation of any point on the curve, that is, to translate the definition into the language of analytic geometry. The equation so obtained *contains implicitly every property of the curve*, and any particular property can be deduced from it by ordinary algebra without troubling about the geometry of the figure. This may have been dimly recognized or foreshadowed by earlier writers but Descartes went further and pointed out the very important facts that two or more curves can be referred to one and the same system of coördinates, and that the points in which two curves intersect can be determined by finding the roots common to their two equations.

In his *Géométrie* Descartes suggests that he had given a method by which algebraic equations of *any* degree could be solved, but in this he was, of course, mistaken. Here also he made use of what we now call "Descartes rule of signs," distinguished between algebraic and transcendental curves, introduced the system of indices now in use, as a^4, and set the fashion of denoting variables by x, y, z, and constants by a, b, c. There seems to be little doubt that the discussion of equations by Descartes was bettered by his knowledge of Harriot's *Ars Analyticae*.

Descartes was the first one to formulate the theorem, commonly attributed to Euler, on the relation between the number of faces, edges, and vertices of a convex polyhedron, $F + V = E + 2$. He did not completely imagine the so-called folium of Descartes,[21] $x^3 + y^3 = 3axy$.

But the greatest French mathematical genius in the seventeenth century was **Pierre de Fermat** who was counselor for the local parliament at Toulouse and devoted most of his leisure to mathematics after he was thirty years of age. He was the founder of the modern theory of numbers and a master in the field of Diophantine analysis.[72] Many theorems, of which he left no proof, have later been proved to be correct. Since it appears that no positive statement which he made has been shown to be incorrect, extraordinary celebrity has been developed in connection with the following theorem [41] of which Fermat stated that he had a proof: No integral values of x, y, z, can be found to satisfy the equation $x^n + y^n = z^n$ if n is an integer greater than 2. For, even to the present day, no mathematician has been able either to prove or to disprove the statement.

The numbers $F_n = 2^{2^n} + 1$ are often called Fermat numbers because Fermat thought that F_n might possibly be a formula for giving an infinity of primes; F_n is prime for $n = 0, 1, 2, 3, 4$, but it has since been shown that it was not prime for 13 other values of n. The numbers F_n became famous through the fact that about 1800

Gauss discovered that when F_n is prime, it represents the number of sides of a regular polygon which can be constructed with ruler and compasses.

Fermat was in possession of the general idea of finding maxima and minima; he also obtained the subtangents and areas of a number of curves, and the center of mass of a paraboloid of revolution.

Fermat seems to have anticipated Descartes in the conception of analytic geometry, and he and Pascal were the founders of the theory of probability,[66] but they published nothing on the subject.

In reviewing Napier's work we found **Cavalieri** enthusiastically advocating the use of logarithms. Cavalieri's chief contribution to mathematics was his principle of indivisibles discussed somewhat by Aristotle and other philosophers and also receiving the attention of Galileo. It occupies [3] a place intermediate between the method of exhaustion of the Greeks, and the calculus methods of Newton and Leibniz. Each indivisible is capable of generating the next higher continuum by motion; a moving point generates a line, a moving line a plane, a moving plane a solid. Though lacking in scientific foundation it was a sort of integral calculus which yielded solutions of difficult problems. A theorem called Cavalieri's [41] theorem has recently come into use in America in the teaching of elementary geometry, as a means of unifying the ideas that lie at the basis of the mensuration of solids. The substance of this theorem is as follows: [53] If two solids have equivalent bases, and if sections parallel to the bases and equally distant from them in the two solids are also equivalent, then the solids are equivalent.

On turning once more to the north we find for our story much of interest in the work of **Christiaan Huygens** eminent as physicist, astronomer, and mathematician, probably the greatest scientist that Holland has produced. Most of his discoveries were made with the aid of ancient Greek geometry, for which, like his friend Newton, he always showed partiality; but at times he called to his aid ideas suggested by Descartes, Cavalieri, and Fermat. Newton and Huygens were kindred spirits and always had the greatest admiration for one another.

Archimedes secured his approximation to the value of π correct to two places of decimals by considering inscribed and circumscribed polygons of $96 = 3.2^5$ sides. Using the same method Viète calculated the value correctly to five places of decimals by means of polygons of 3.2^{17} sides, and Ludolf van Ceulen to 20 places by means of polygons of 60.2^{29} sides; after spending most of his life in calculation he used this same method to determine the result to 35 places of decimals. In a remarkable work on the magnitude of a circle Huygens showed, after a series of propositions which he proved geometrically, that Archimedes's value could be got by considering

only a triangle, and a value correct to 9 places by considering only a hexagon.[20]

Huygens wrote also the first formal treatise on probability, which was the best until superseded by more elaborate works of Jean Bernoulli and DeMoivre. To the theory of curves he added the important theory of evolutes and showed in particular that the evolute of a cycloid is an equal cycloid, which led to his construction of the isochronal pendulum. Stimulated by the astronomer's need, he invented the pendulum to regulate the movement of clocks. Huygens showed also that the cycloid has the property that when the accelerating force is gravity a particle moving along a vertical curve will arrive in the same time at a given point wherever the initial point is taken on the curve. Huygens solved the problem of determining the equation of the catenary curve formed by a perfectly flexible chain suspended by its ends. He found also many other results concerning curves, some of them generalizations of particular cases considered by Fermat, and dealt with many interesting problems of pure geometry.

In England of the seventeenth century the work of Harriot and of Oughtred has been considered; but a much abler mathematician, one of the most original of his day was **John Wallis,**[41] for 54 years professor of geometry at Oxford University. He was a man of great erudition, and wrote works in a dozen different fields. His great work on algebra contains the first record of an effort to represent an imaginary number graphically by the method now used. He connected the problems of finding the area and length of a curve by obtaining the equivalent of the formula $ds = dx[1 + (dy/dx)^2]$. He used Cavalieri's method of indivisibles to find the areas of curves $y = x^m$ when m is a positive integer or a fraction. He failed to get the approximate quadrature of the circle $y = (1 - x^2)^{1/2}$ by series, since he did not know the binomial theorem; but by another method he expressed π in the form of an infinite product,

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdot \frac{8}{9} \cdots,$$

thus bringing us to the second period in the history of the problem of squaring the circle,[20] the first having ended with discussions of polygon perimeters.

But the greatest scientist of the seventeenth century was **Isaac Newton,**[54–57] who was born in 1642 and died in 1727, aged 85. Already at Cambridge University, at the age of 23, in the year that he took his B.A. degree, he had invented the method of fluxions [41] or calculus, and discovered the binomial theorem.[41] In the following year he made applications of the calculus to tangents, curvature,

concavity and convexity, maxima and minima, and other things, some of which were new and of importance. It was about this time also that he made interesting discoveries in optics and formulated, but did not announce, the law of universal gravitation.[55] Referring to these years later in life Newton wrote, "in those days I was in the prime of my age for invention, and minded Mathematicks and philosophy more than at any time since."

His former teacher **Isaac Barrow,** fully recognizing the transcendent ability of Newton, resigned his chair (although only thirty-nine years old) to make way for Newton then only twenty-six years of age to be appointed in his place as professor of mathematics at Cambridge. When forty-seven he represented the University in Parliament and by the time he was fifty his scientific creations were practically ended; but in later years he received many honors and a number of his earlier writings or lectures were published.

His greatest work was indisputably the "Mathematical Principles of Natural Philosophy" usually known by a single word of the Latin title, *Principia*, published in 1687. This work [54] (1) treats of the motion of particles or bodies in free space either in known orbits, or under the action of known forces or under mutual attractions; (2) treats of motion in a resisting medium, and of hydrostatics and hydrodynamics with special applications to waves, tides, and acoustics; and (3) applies theorems obtained to the chief phenomena of the solar system and determines the masses and distances of the planets. The motions of the moon, the various inequalities therein, and the theory of tides are worked out in detail. The great principle underlying the whole work is that of universal gravitation characterized as "indisputably and incomparably the greatest scientific discovery ever made." In order that his work might be more easily understood by students of his time, Newton achieved the stupendous task of translating his arguments into the language of Greek geometry even though many of his results were first worked out by methods of the calculus. In the *Principia* are a great many results concerning well-known curves, such as epi- and hypocycloids, and many beautiful results in geometry, such as the following: (1) If a point P moving along a given straight line is joined to two fixed points O and O', and if lines OQ and $O'Q$ make fixed angles with OP, $O'P$, then the locus of Q is a conic. (2) The centers of the three diagonals of a complete quadrilateral lie on a line which is the locus of the centers of the conics tangent to the sides of the quadrilateral.

Newton studied curves of the third order with a view, apparently, to illustrating the application of analytic geometry to curves other than conics. He enumerated 72 of the possible 78 forms which a cubic may assume, in his classification; and in the course of the work

he states the remarkable theorem that just as the shadow of a circle cast by a luminous point on a plane gives rise to all the conics, so the shadow of the curves represented by the equation $y^2 = ax^3 + bx^2 + cx + d$ gives rise to all cubics. This remained an unsolved puzzle till a proof was discovered in 1731.

In another monograph on the quadrature and rectification of curves by means of infinite series, Newton indicates at one point the importance of determining whether the series are convergent—an observation far in advance of his time—but he knew of no general test for the purpose; and in fact it was not until Gauss and Cauchy took up the question, early in the 19th century, that the necessity for such limitation was commonly recognized. In this monograph arcsin x is expanded in a series, and the process of reversion of series is employed.

In his *Universal Arithmetic*, containing the substance of lectures, Descartes's rule of signs is extended to give limits to the number of imaginary roots of an equation, and a method is given for finding the approximate values of the roots of a numerical equation,[58] a method which applies equally to algebraic or transcendental equations. He enunciated also the theorem, known by his name, for finding the sum of the nth powers of the roots of an equation, and laid the foundation of the theory of symmetric functions of the roots of an equation.

In connection with various challenges among mathematicians, Newton was never beaten; in one from Leibniz he solved the problem of finding the orthogonal trajectories of a given curve. One of the most interesting eulogies on Newton's work is that attributed to Leibniz who upon being asked by the Queen of Prussia what he thought of Newton, his answer was: "Taking mathematicians from the beginning of the world to the time when Newton lived, what he had done was much the better half." [46]

Leibniz, the only pure mathematician of the first class produced by Germany during the seventeenth century, was equally notable in the field of philosophy. The last years of his life were embittered by a controversy as to whether he had discovered the calculus independently of Newton's previous investigations,[59] or had derived the fundamental idea from Newton. There is now no doubt that Leibniz developed his calculus quite independently, and that he and Newton are each entitled to credit for their respective discoveries. The two lines of approach were radically different, although the respective theories accomplished results that were practically identical. Leibniz was original in much that he did. He made the Cavalieri method scientific. He introduced the idea of fractional differentiation.[59a] He laid the foundation for the theory of envelopes and defined the osculating circle, showing its importance in the study of curves.

The history of determinants [60] begins with Leibniz in connection with the eliminant of linear equations. He gave a generalization of Newton's binomial expansion rule in his expansion of any polynomial $s + x + y + z + \cdots$ to any arbitrary power r. We have already referred to the calculating machine [41] he constructed. He derived the series arctan $x = x - (1/3)x^3 + (1/5)x^5 - \cdots$, from which $\pi/4 = 1 - 1/3 + 1/5 - 1/7 + \cdots$, but both had been found before.

By making use of this expansion for arctan x in the formula $\pi/4 = 4$ arctan $1/5 -$ arctan $1/239$, discovered by an eighteenth century English mathematician, another English mathematician named William Shanks in 1874 calculated π correct to 707 places of decimals.

Leibniz made important contributions to the notation of mathematics. Not only is our notation of the differential and integral calculus due to him but also our signs, in geometry, for similar ($\sim$), and for equal and similar, or congruent, ($\simeq$).

And finally, among the names of mathematicians of the last half of this century we shall briefly note results found by **Jacques Bernoulli** of Switzerland, a member of a remarkable family in which 8 members distinguished themselves in mathematics. He was professor of mathematics at the University of Basel, and was interested in the fields of astronomy, mathematics, and physics. He was the first to solve the problem of isochronous curves proposed by Leibniz, that is, to find the curve along which a body falls with uniform vertical velocity. Huygens also solved this problem and found it to be the semi-cubical parabola [21] $x^3 = ay^2$.

He worked on the problem, To find the curvature a lamina should have in order to be straightened out horizontally by a weight suspended at one end. He noted the relation $Rs = a^2$, where R is the radius of curvature at a distance s along the lamina from the point of suspension, and gave a construction for points of the curve. But it was Euler in the next century, who first really visualized the curve as a double spiral [61] with asymptotic points, later called the clothoïde; [21] the equation of the spiral is:

$$x = \frac{a}{2^{1/2}} \int_0^v \frac{\sin v dv}{v^{1/2}}, \qquad y = \frac{a}{2^{1/2}} \int_0^v \frac{\cos v dv}{v^{1/2}}.$$

This is the spiral which comes up in connection with Fresnel's discoveries of the diffraction of light, and in railway easment curves. It has a number of interesting properties.

Bernoulli did much work on the logarithmic spiral [62] and found for example that (1) the evolute of a logarithmic spiral is another equal logarithmic spiral, (2) the pedal of a logarithmic spiral with respect to its pole is an equal logarithmic spiral; (3) the caustic by reflection and refraction of a logarithmic spiral for rays emanating

from the pole as a luminous point is an equal spiral. Such perpetual renascence delighted Bernoulli and it was, in accordance with his directions, inscribed on his tomb with the inscription: *Eadem mutata resurgo*—"I arise the same though changed."

Bernoulli wrote also a posthumously published work entitled *Ars Conjectandi* which established principles of the calculus of probability [66] in more elaborate form than Huygens had done. It is here that certain numbers come up, and are known in a very extensive later literature as "Bernoulli's numbers."

Thus closes our sketch of the seventeenth century during which such great advances were made in fields of analytic geometry, calculus and its applications, mechanics and dynamics, geometry, theory of probabilities and the theory of numbers.

B. The Eighteenth Century

The principal mathematicians flourishing in this century were, DeMoivre, Taylor, Maclaurin, Euler, D'Alembert, Lagrange, Lambert, and Monge.

Since **Abraham DeMoivre,**[63] although born in France, spent 96 years of his life in England, he is properly considered a member of the English school.[30] He was an intimate friend of Newton. Apart from numerous memoirs he published a work called *Annuity upon Lives*, of which there were seven editions, a work of importance in the history of actuarial mathematics. His very valuable *Doctrine of Chances* passed through three editions and contained much new material connected with the theory of probability; [66] indeed, he ranks with Laplace in making the most important contributions to the subject up to the end of the nineteenth century. He gave the first treatment of the probability integral,[64] and essentially of the normal curve.[65] He formulated and used a theorem improperly called Stirling's theorem.[64]

In his *Miscellanea Analytica*,[66] 1730, which brought about his election as a Fellow in the Berlin Academy, there is important material in connection with recurrent series, analytic trigonometry involving imaginaries, and problems of chance, and analysis is applied to problems of astronomy. From one part of this work it is clear that DeMoivre was thoroughly familiar with the formula now connected with his name: [41]

$$(\cos x + i \sin x)^n = \cos nx + i \sin nx,$$

where n is an integer. But the result is not explicitly stated in any of DeMoivre's writings; a form equivalent to this was, however, given in 1748 by Euler.[41] In this work DeMoivre studied also the quadratic factors of $x^{2n} - 2x^n \cos n\theta + 1$ and the geometrical interpretation of these factors as the squares of the distances of a

certain point from the vertices of a regular polygon of n sides, "DeMoivre's property of the circle," is now given in most works on analytic trigonometry.

The name of the Englishman **Brook Taylor** is familiar to every student of the calculus, through the theorem of Taylor's expansion of $f(x + h)$, given in his *Methodus Ivcrementorum directa et inversa.* This is the first treatise dealing with finite differences; it was published in 1715. The convergency of the expansion is not considered.

In this work Taylor solved the following problem which he believed to be new: "To find the number of vibrations that a string will make in a certain time having given its length, its weight, and the weight that stretches it." In discussing the form of the vibrating string, his suppositions regarding initial conditions, including that it vibrated as a whole, led to a differential equation whose integral gave a sine curve. Thus started a discussion which was to culminate a century later in the work of Fourier.[17]

We shall next consider **Colin Maclaurin,**[67] a Scot, and one of the ablest mathematicians of the eighteenth century. His name is associated with Taylor in the expansion of $f(x + h)$, when $h = 0$. This was given in his *Treatise of Fluxions* in 1742 but the same thing appeared twelve years earlier, in a publication by Stirling. Maclaurin's *Treatise* is especially valuable for the solutions it contains of numerous problems in geometry, statics, the theory of attractions, and astronomy. He followed Newton in using the geometrical method of analysis. Lagrange characterized the work as "a *chef d'oeuvre* of geometry which one can compare with all that which Archimedes has left as the most beautiful and most ingenious." By avoiding the use of analysis and of the infinitesimal calculus he helped to perpetuate methods [68] not abandoned in England till about 1820.

When only 21 years of age Maclaurin wrote a work called *Geometria Organica* [69] which was inspired by Newton's work on curves of the third order, and by the organic description of a conic which we quoted from Newton's *Principia.* He attempted to generalize this organic description of a conic so as to obtain curves of all possible degrees by a mechanical description. He discusses also various pedal curves.[21] In this way he was led to such familiar curves as the cissoid, strophoid, cardioid, limaçon, and lemniscate, and many others [21] which have since become notable in connection with the study of special curves.

We have already more than once referred to **Léonard Euler** [70] the great Swiss mathematical genius, also born at Basel, the home of the Bernoullis. Euler was born in 1707. The project of erecting an Academy at Petersburg, which had been formed by Peter the Great,

led to Euler being invited in 1733 to the chair of mathematics there. After eight years he accepted the invitation of Frederick the Great to go to Berlin to direct the Prussian Academy which was then rising in fame; he continued there for twenty-five years, till 1766, when he returned once more to Petersburg, where he remained till his death in 1783 at the age of 76.

Few writers in mathematics ever contributed so extensively or so fruitfully. In a sense he was the creator of modern mathematical expression. He revised almost all the branches of mathematics then known, filling up details, adding proofs and new results, and arranging the whole in a consistent form. Close upon 800 memoirs and papers came from his hand,[71] besides many treatises. He wrote on philosophy, astronomy, physics, geography and agriculture; and on mathematics in the following fields, among others: arithmetic, theory of numbers, Diophantine equations,[72] algebraic equations, series, continued fractions, calculus of probability,[66] analysis, definite and indefinite integrals, elliptic functions and integrals,[73] differential equations, calculus of variations, elementary geometry, analytic geometry, differential geometry, mathematical recreations, music,[17] mechanics, ballistics, and the motion of ships.

Certain of our ordinary notations originated with Euler,

$f(x)$ for function symbol,
e for $2.71828\cdots$,
s for half the sum of the sides of a triangle,
a, b, c for the sides of a triangle ABC,
$\sum$ for summation sign,
i for $\sqrt{(-1)}$.

From the formula $e^{ix} = \cos x + i \sin x$, for $x = \pi$, $e^{i\pi} + 1 = 0$. This relation was to be the basis of the proof that the problem of squaring the circle was impossible. It should be noted that thirty years before Euler published his result Roger Cotes gave the equivalent in words of the relation:

$$ix = \log (\cos x + i \sin x).$$

Euler showed that $\pi/2 = -i \log i$, and that $(\log i)/i$, and i^i had an infinite number of values. He gave the formula, $\pi = 20 \arctan 1/7 + 8 \arctan 3/79$, and from it calculated π to 20 places of decimals.

Euler showed that the center of the circumscribed circle of a triangle, its center of gravity, and the point of meeting of the altitudes are collinear;[74] hence the name *Euler's Line*. It was later shown that the center of the nine-point circle is also on this line.

Euler gave a new method of solving the general equation of the fourth degree. He introduced the idea of an indicatrix for studying the variations of the radii of curvature of normal sections of a

surface at a point. He worked on the problem of a vibrating string [17] initiated by Brook Taylor, and later carried on by Lagrange, preparatory to the important work of Fourier in the early part of the 19th century. It was in his work on the calculus of variations that Euler considered the problem of Bernoulli for a coiled lamina,[61] to which reference has been made. Beta and gamma [75] functions originated with Euler, as well as the Euler constant

$$\text{Lim } (1 + 1/2 + 1/3 + \cdots + 1/n - \log n) = 0.5772 \cdots.$$

When Euler left Berlin in 1766, he was succeeded by **Joseph Louis Lagrange** [26] an Italian by birth, a German by adoption and a Parisian by choice,[4] one of the great mathematicians of all time. He was then 30 years old. In sending the invitation Frederick the Great wrote that the greatest king in Europe wanted the greatest mathematician of Europe at his court. Lagrange remained there till the death of Frederick 21 years later. Many of his great memoirs, as well as the composition of his monumental work on *Analytic Mechanics* (containing his well known general equations of motion of a dynamical system), date from this period. During the remaining 25 years of his life he was in Paris, a professor at the newly established École Normale Supérieure, and École Polytechnique. These two great schools occupy a most important place in the history of mathematics. Most of the greatest mathematicians of France have been professors in them, and practically all have there been trained. Towards the close of the eighteenth century appeared two more of Lagrange's great works, the one, the *Resolution of Numerical Equations of Every Degree*, and the other *Theory of Analytic Functions containing the Principles of the Differential Calculus*. In the first of these he gave the proof that every algebraic equation has a root, a result earlier regarded as self-evident. His work in the field of Calculus of Variations was important and his name is linked with that of Euler in founding the subject. His contributions in the field of Differential Equations were notable. Like Diophantus and Fermat he had a special genius for the theory of numbers. As a whole his work had a very profound influence on later mathematical research.

Let us now turn back to Germany to consider contributions made by **Johann Heinrich Lambert** who enjoyed association with Euler and Lagrange in Berlin. He was a many-sided scholar and wrote on a great variety of subjects.

His *Freye Perspective* of 1774 contained numerous geometrical problems of interest.

In a monograph of 1761, on the determination of a comet's orbit, he gave incidentally the results that a circle about a triangle circumscribing a parabola goes through the focus; and hence if four

tangents are given the focus is determined. From the rediscovery of this parabola property, about forty years later, **Wallace** deduced his theorem, The feet of the perpendiculars on the side of a triangle, from any point on its circumscribed circle, are collinear.

It was in 1761 also that Lambert proved, with most minute rigor, that the numbers π and e were irrational. This was done (with the aid of a result by Euler) by expressing $\tan z$ as a continued fraction in z. This result was the beginning of the third period in connection with the settling of the problem of the squaring of the circle.[20] In 1844 and 1851 a prominent French mathematician named **Liouville** proved the existence of transcendental numbers, that is of numbers which could not be found by ordinary algebraic operations on integers. In 1873 a great French mathematician named **Hermite** proved that e was transcendental.[41] As a result, and by means of Euler's relation $e^{i\pi} + 1 = 0$, it was shown, in 1882, that π was transcendental, and hence that the problem of squaring the circle was impossible of solution with ruler and compasses.[76] This last result was found by Lindemann, a German.

To Lambert we owe the systematic development of the theory of hyperbolic functions, introduced by Vincenzo Riccati a few years earlier.[77] Lambert's name is also associated with important projections used in map making.[78]

Three other French mathematicians of prominence, Monge, Laplace, and Legendre, were born about the middle of the eighteenth century, but the culminating period of the most important work of the two latter was in the nineteenth century. For the final name in the eighteenth century, we shall, then, consider that of **Gøsperd Monge,**[79] a professor of mathematics [80] in the École Polytechnique of Paris. He was the inventor of descriptive geometry,[41] and his treatise on this subject, which appeared first in 1794, went through a number of editions. There were five editions of his *Applications of Analysis to Geometry* which first appeared in 1795; it was an important early work on the differential geometry of surfaces.

Our survey has suggested that while the eighteenth century saw the further development of such subjects as analysis, theory of numbers, differential equations, probabilities, trigonometry, and analytic mechanics, it saw also the creation of new subjects such as actuarial science, calculus of variations, finite differences, gamma and elliptic functions, descriptive geometry, and differential geometry. We note also that practically all mathematics used by the American engineer was developed before the nineteenth century.

C. The Nineteenth Century and Later

Moritz Cantor's great history of mathematics to the end of the eighteenth century contained four large volumes with an average of

988 pages to the volume. It has been estimated that if the history of mathematics in the nineteenth century were written with the same detail it would require fourteen or fifteen more similarly sized volumes.[81] As indicated in our introductory remarks we shall, in conclusion, touch very briefly on only a very few topics in the enormous mass of available material.

We saw that Laplace and Legendre were two great French mathematicians who were contemporaries of Lagrange, but whose principal work was published in the nineteenth century. Let us now consider them. **Pierre Simon, Marquis de Laplace,**[82] born in obscurity, became titled through political activities. He participated in the organization of the École Normale Supérieure and École Polytechnique. His researches were carried on chiefly in the fields of astronomy, celestial mechanics, probabilities,[66] calculus, differential equations, and geodesy. In 1796 he published a nonmathematical popular treatise on astronomy entitled, *Exposition of the system of the world.* It was followed by his monumental mathematical *Celestial mechanics,* embracing all the discoveries, in this field, of Newton, Euler, Lagrange, and others who preceded him, as well as his own. This work stamped him at the time as an unrivalled master in the field. (It was translated into English with elaborate commentary by Nathanial Bowditch, the New England author of the *New American Practical Navigator,* which has passed through more than 65 editions since the first in 1799.) So also, in 1812–14, Laplace published both popular and mathematical treatises on probability. In the mathematical work he was by no means particular to credit predecessors with their discoveries. It may well be, then, that Laplace knew that DeMoivre had treated the probability integral, as well as what is essentially the normal curve, long before him.[64]

The notion of a potential function introduced into analysis by Lagrange was much used by Laplace who showed that if V is such a function it satisfies the partial differential equation:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0,$$

called Laplace's equation.

Adrien Marie Legendre is chiefly known in higher mathematics for important works on the theory of numbers, on elliptic functions and integrals,[73] the result of 30 years of meditation,* and on exer-

* Legendre's work appeared almost simultaneously with the complete theory of elliptic functions of the great Norwegian mathematical genius **Abel,** which Legendre recognized as a great advance over his own. See G. Mittag-Leffler, *Niels Henrik Abel,*[41] Paris, 1907 (reprint from *La Revue du Mois*), pp. 34–38. Legendre was similarly impressed with the work of **C. G. J. Jacobi** which appeared somewhat later (1829).

cises in the integral calculus (3 vols.). Most of these works deal with topics in higher mathematics the discussion of which is beyond the scope of this sketch. Legendre was the first one to publish the method of least squares.[33, 41] Legendre wrote also a work on the elements of geometry and trigonometry, first published in 1794, which has gone through many editions. It sought to rearrange the propositions of Euclid separating the theorems from the problems and simplifying the proofs, without lessening the rigor of the ancient methods of treatment. To this work is largely due the abandonment of Euclid as a text-book in American schools. In successive editions Legendre tried to prove the fifth postulate of Euclid and every student of the foundations of geometry takes account of results thus found.[22] The work contains also a proof that both π and π^2 are irrational, a proof of the first result having been already given by Lambert.

A geometry which is built up without assuming Euclid's fifth postulate as true is said to be non-Euclidean.[41] Such a geometry was worked out almost simultaneously by a Russian and a Hungarian. The Russian, **Nicolai Lobachevsky,**[84] studied at the University of Kazan and became a professor of mathematics there in 1814 at the age of twenty-one. In 1826 he made known, through his lectures, his conception of such a geometry which should not depend upon the Euclidean postulate of parallel lines. These ideas were published in 1829 and later. The Hungarian, **Farkas Bolyai,**[85] whose results were worked out in 1825–26, published them in 1832. The results thus presented were, later in the century, developed into a body of theory of high importance in mathematics.[86]

The fundamental ideas of non-Euclidean geometry were also at this time familiar to **Carl Friedrich Gauss,** who spent most of his life at Brunswick or Göttingen, Germany. From a child of two years detecting an error in his father's calculation of the wages of an employee, he became the greatest mathematician that Germany, and one of the greatest mathematicians that the world, has ever known. He was born in 1777 and was led to take up the study of mathematics by his discovery in 1796 that it was possible to construct a regular polygon of 17 sides by means of a ruler and compasses. He was later able to show that when the Fermat number $F_n = 2^{2^n} + 1$ is prime, it represents the number of sides of a regular polygon constructible with ruler and compasses.[41] He did not prove, however, what is a fact, namely that these are the *only* polygons, the number of whose sides is prime, which are so constructible. Only four such polygons are known at present. This discussion occupied a very small part of Gauss's greatest single work, *Disquisitiones Arithmeticae,* which deals with matters of fundamental importance in the theory of numbers.[41] His memoirs

of 1825–27 on general investigations of curved surfaces, fundamental in differential geometry, were translated into English, edited with a bibliography showing resulting developments, and published by Princeton University (1902).

Gauss worked and created in many other fields, astronomy, geodesy, electricity, complex numbers, least squares, and almost every leading branch of mathematics. He gave the first wholly satisfactory proof that every algebraic equation of degree m has exactly m roots[41] (which Gauss calls the Fundamental Theorem of algebra), in his doctoral dissertation written when he was twenty years of age. He here shows defects in the earlier consideration of this question by d'Alembert (1746), Euler (1749), Foncenex (1759), and Lagrange (1772). Gauss gave three other proofs of this theorem in 1816 and 1850. He died in 1855 aged 78.

Gauss was a notable instance of a youthful arithmetical prodigy who later achieved greatness in mathematics. The most extraordinary mental calculating prodigy who ever lived was **Zacharias Dase**[88] of Hamburg who died in 1861, aged 37. He multiplied two 8-figure numbers in 54 seconds, two 20-figure numbers in 6 minutes, two 40 figure numbers in 40 minutes and two 100-figure numbers in $8\frac{3}{4}$ hours, and found the square root of a hundred figure number in 52 minutes. Gauss tried to turn his abilities to some useful purpose, and as a result we have Dase's seven-place tables of the natural logarithms of numbers from 1 to 10,500 (1850) and his factor tables of all numbers in the sixth, seventh, and eighth million (1862–65).

In considering the mathematics of the seventeenth century we referred to the notable development of synthetic geometry by Pascal and Desargues. There was practically no extension of their work until the early part of the nineteenth century when enormous strides were made by the Frenchman **Victor Poncelet** and the Swiss **Jakob Steiner,** who spent most of his life in Berlin. It was Poncelet who first showed that every construction with ruler and compasses can be carried through with ruler alone if a circle and its center are given in the plane of construction. Very early in the nineteenth century the principle of duality was formulated, and **Brianchon,** a student in the École Polytechnique, enunciated the dual of Pascal's theorem, namely, The three diagonals of a hexagon circumscribing a conic are concurrent.[41]

We have referred to the proof of the impossibility of the problem of squaring the circle. The other two famous problems of the ancients lead to cubic equations. In 1837 these problems also were shown to be impossible of solution with ruler and compasses, when it was proved[19] that the corresponding cubic equations were irreducible in the Euclidean domain.

The transformation of figures in the plane by circular inversion is of importance in both pure and applied mathematics. This seems to have been first conceived by Steiner in 1824, and by the Belgian astronomer and statistician Adolphe Quetelet in the following year. German, Italian and Irish expounders of the method soon followed.

In the nineteenth century the question of discovering a linkage for drawing a straight-line was worked on considerably.[3] A beautiful arrangement of seven jointed rods was finally devised in 1864 by a Frenchman named **Peaucellier.**[41] The construction of this was based upon the principle that the inverse of a circle through the center of inversion is a straight line. The same principle was used later in a new straight line linkage containing only five links. Many linkages were found [39] for constructing special curves, such as conics, cardioids, lemniscates, and cissoids. It has been proved that a linkage is possible for tracing every algebraic curve, but that it is not possible to construct a linkage for tracing any transcendental curve. Linkages have also been used for solving algebraic equations.

We have seen that the solutions of the general equations of the third and fourth degrees were found in the sixteenth century by Tartaglia and Ferrari. Nearly three hundred years were to elapse before it was shown that a similar solution of the general equation of the fifth degree was not possible. This was proved by the Norwegian genius **Niels Henrik Abel** [41] in a pamphlet which he published in 1824. We have already referred to Abel's priority in connection with elliptic functions.

Of methods for approximating to the real roots of numerical equations we noted that of Newton by successive substitutions in derived equations.[58] This dates from 1669 although it was first published in Wallis's *Algebra* in 1685. It is the method which to this day has held undisputed sway in France. The so-called "Horner's method," really given fifteen years earlier by an Italian **Paolo Ruffini,** was in a paper by **William G. Horner,** read before the Royal Society of London in 1819, and published in the same year. It soon became widely used in England, and later in the United States, and to a less degree in Germany, Austria, and Italy. A valuable work on numerical equations by Fourier was published in 1831, after his death; but for nearly twenty-five years before this he had taught his students at the École Polytechnique the theorem now associated with his name,* concerning the limit to the number of real roots in any given interval. In 1829 **Charles Sturm** stated a theorem for determining the exact number of roots in any interval; this was merely a by-product of his extensive investigations of linear

* First published by Fourier in 1820. The name of **Budan** is also sometimes associated with this theorem.

difference equations of the second order,[90] although it was suggested by a close study of the manuscript of Fourier's work of 1831. Considerable work has been done also in deriving the roots of numerical equations by means of infinite series.[58]

The problem of determining the imaginary roots of numerical equations has been considered by many writers. Sturm's theorem gave a method for determining the number of such roots, but not their location. This was revealed by a general theorem of a great French mathematician **Augustin Louis Cauchy** (1789–1857), giving the number of roots real or complex which lie within a given contour.

A few comments in connection with the convergency of infinite series,[2] so necessary for consideration by every mathematician, are now appropriate. We noted that this question was discussed by Newton in a particular case. But the first important and strictly rigorous investigation of infinite series was made by Gauss in 1812 in connection with hypergeometric series. In 1821, however, Cauchy gave a discussion and tests of general application, such as are now constantly in use. A paper of Abel on power series, published in 1826, is also of importance in this connection.[41] The convergence of trigonometric series was also naturally considered about this time. We have seen that problems of Taylor and Euler in connection with a vibrating string called for trigonometric series. So also in other connections such series arose in works of Bernoullis, Legendre, and Laplace. But Fourier (1812–22) was the first to assert and to attempt to prove that any function, even though for different values of the argument it is expressed by different analytical formulas, can be developed in such a series. Since the real importance of trigonometric series was thus shown by Fourier for the first time it was natural that his name should thereafter be associated with them. For information as to the application of such series to many problems of mathematical physics the sketch of Bôcher [91] may be consulted.

Vector analysis as employed today is largely due to two mathematical physicists, **J. W. Gibbs,**[92] an American, 1881–84, and **Oliver Heaviside,**[93] an Englishman, 1891. Quaternions, an algebra of vector magnitudes, was developed earlier, about 1853, by that Irish genius, **W. R. Hamilton.**[94]

We have already noted that the subject of determinants originated with Leibniz. Laplace, Lagrange, Gauss, Cauchy, Jacobi, and many others contributed to the development of the subject before 1841, as indicated in the first volume of Thomas Muir's monumental work.[60] Four other volumes of a similar size are necessary for setting forth the history of the subject during the next eighty years.[60]

In the course of my remarks I have referred incidentally to the great tables of three of the natural trigonometric functions by Rhaeticus; to the tables by Briggs and to their tremendous elaboration by the Pearson Laboratory in England; to factor tables and tables of the natural logarithms by Dase; and to works of Legendre containing valuable tables of elliptic integrals. Such remarks might readily be greatly multiplied. Indeed much more than an hour might be spent in giving the history of the vast number and variety of mathematical tables which have been prepared.[95]

The history of the tables of logarithms of numbers would unfortunately have to expose the fact that most of the ten-place table which **W. W. Duffield** published in the *U. S. Coast and Geodetic Survey, Report*, for 1895–96, was taken without acknowledgement from Vega's fine tables of a century before. But, on the other hand, the history would point with pride to the superb factor tables and table of primes of **Derrick Lehmer,** replacing combined works of **Burckhardt, J. Glaisher,** Dase, and others. The history of the century-long series of tables of the hyperbolic functions would also include those of the Smithsonian Institution by **G. F. Becker** and **C. E. Van Orstrand,** and of the University of California by **F. E. Pernot** and **B. M. Woods.** Familiarity with the lengthy history of tables of natural trigonometrical functions, culminating in the magnificent work of **Andoyer,** of all six functions for every ten seconds arc of to 14–17 places, would probably have prevented the publication and sale in this country of inferior excerpts or adaptations from them. Not even reviewers of the recent New York volume of tables of $\log (a + b)$ and $\log (a - b)$, where $\log a$ and $\log b$ are known, first given by **Leonelli** and Gauss, seemed to be familiar with the fact that such tables were readily available before.

In England a splendid elaborate volume which contains the history of tables of logarithms of numbers has been recently published.[96] In Germany a pamphlet guide to various tables of elliptic integrals, Bessel's functions, and spherical harmonics, has been available for several years.[97] But a great work yet remains to make a vast body of most important fundamental material readily available for the engineer and general research work. This has been recently recognized by our National Research Council.

In conclusion we may indicate some references to historical accounts of developments of higher mathematics. The sketch of E. W. Brown [98] sets forth the history of mathematics to the end of the second decade of the twentieth century. The development of applied mathematics in the nineteenth century is surveyed by R. S. Woodward.[99] J. W. Nicholson [100] attempted to show the directions in which progress in pure mathematics was actually being made a decade ago, or in which progress might be hoped for shortly thereafter.

I end these very imperfect lectures with a thought even more true to-day than when put into writing nearly three centuries ago.

> "The Mathematics . . . is the fruitful Parent of . . . Arts, the unshaken Foundation of Sciences, and the plentiful Fountain of Advantage to Human Affairs."

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THE TEACHING OF TRIGONOMETRY FOR ENGINEERS *

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Trigonometry is essentially a very simple subject and a comparitively small part of the mathematical curriculum. There are several reasons, however, why I am glad to discuss it in particular. In the first place, its very brevity enables me to give attention to every phase of the subject. Moreover, I am free to digress somewhat into the more general questions of the reasons for the teaching of any mathematical course. This gives me the opportunity to open this Summer Session, in so far as the separate courses in mathematics are concerned, with a comprehensive statement of the aims of mathematical instruction which will, I hope, act as a sort of keynote for this whole Session.

Again, trigonometry offers particular opportunities for the illustration of principles that should be emphasized throughout mathematics. The function idea, which is the central theme of all mathematics, appears in simple but unmistakable form. The idea of motivation can be illustrated simply, as well as the absence of motivation in some topics. The question of possible transfer of training can be illustrated, both in the positive and in the negative sense. Usefulness, tool-value, training for comprehension, training for skill, values and dangers of memorization, drill-work and its limits, can all find a place in this discussion.

Aims of Teaching Trigonometry

General Aims of Mathematics. The general aims of mathematical instruction have been discussed thoroughly and well by the National Committee on Mathematical Requirements, a committee organized by the Mathematical Association of America. Its report is in every university and college library, and should be familiar to all teachers of mathematics.

a. A central aim which I may stress is the control of quantitative relations. That quantitative relations exist in the world about us, and enter into the lives of all active people, will be admitted by all. If we shape our courses toward emphasis upon this theme, as we well

* A lecture delivered at the Mathematics Session of the Summer School for Engineering Teachers, University of Minnesota, August 26, 1931.

may, much of the common criticism of mathematics in the school system will automatically fall to the ground. If we do not, if we omit to stress this theme, mathematics will suffer, and our student will not receive the maximum possible benefits from our courses.

b. Strongly allied to the preceding topic is the idea of formation of the habit of functional thinking. This again has very close relations to the work of the world, and enters into the lives of all active persons. To consider in every quantitative situation what other quantities affect that in which we are interested, and how, may well become a fixed habit, so that the individual will automatically consider this question when confronted by any quantity which affects him.

c. The National Committee has called attention to the tendency toward making mathematics purely mechanical, and has pointed out that this interferes with the attainment of the aims just mentioned. Drill on manipulation should be limited to the amount necessary to a complete understanding of the principles involved. To carry drill beyond the point of complete understanding of all the principles is to lose ourselves in the "disciplinary" grind of work for work's sake, an idea which has long since been abandoned in every reputable educational discussion.

d. The acquisition of specific skills, however, is not to be abandoned, when those specific skills are needed by the student for specific ends. Mathematics can serve in many instances for direct training in a skill that is needed by the student. For prospective engineers, such needs are very evident, even to the student himself. Mathematics as a tool for engineering and for other subjects seems to some a degradation of the subject: to me the necessity for some part of mathematics in any walk of life seems a very welcome thing. It gives point to our courses, it inspires that interest which Professor Spaulding has so well emphasized. Indeed he has told us that we should on every occasion and in each topic search diligently to find these uses in other subjects, and dwell upon them. I welcome then any tool-value which we can discover; I take pride in mathematics as a tool for engineering.

Application to Trigonometry. The application of the aims which I have mentioned to the subject of trigonometry is so clear that it may well be a typical case for any such discussion.

a. The existence of quantitative relations in triangles is evident to all who begin a study of it. That such relations do exist appears even before the functions commonly used are defined, for the fact of determination of the angles by the sides, for example, appears from mere drawing. The usefulness to engineers and others is also apparent to all students, so that the fundamental elements for interest and for the quantitative relationships exist from the start. These

should never be forgotten, nor should some lesser aim be allowed to obscure them.

b. The opportunity for functional thought, and the formation of habits of functional thinking are also obvious. What is the effect of changing the length of one side of a triangle? How does the tangent of an angle change near zero; near ninety degrees? These only serve to illustrate the many functional questions which present themselves—if the teacher does not exclude them—at every point.

The traditional functions are not linear, nor indeed algebraic. They therefore increase very much the range of functional thought of the student. The ideas of maxima and minima occur naturally. Periodicity occurs for the first time in the student's experience. Tables occur here (and in the allied study of logarithms) as a serious means of functional representation. The effect of errors of observation upon the computed parts is a functional idea of great importance which is new to most students at this time. In all, a wide variety of functional thinking is presented, if it is not suppressed, so that the subject may contribute very markedly toward the formation of good habits of functional thought.

c. Excessive drill is entirely possible, and it should be avoided. Not so much in the solution of triangles, where the formation of definite skills is needed, does this danger arise. In triangle solutions, our habits are probably not at all excessive. In other topics in trigonometry, however, excesses of manipulative drill do exist. Thus under the head of analytic trigonometry, the student is called upon to prove the truth of trigonometric identities; this work is often carried beyond the bounds of complete understanding of the principles and into the realm of purely "disciplinary" work for work's sake. Such drill as this, in a topic in which a skill is not required for any need of the engineer, is a waste of time when it goes beyond the bounds of complete understanding of the principles involved. The same thing may be said in the case of drill in the solution of trigonometric equations. Such equations occur very little, and no skill is to be established. Hence drill should stop when the principles involved are fully understood.

d. For engineers, as I have said, certain skills are needed. The solution of triangles is chief among these, and I have justified all the drill that is now common on that topic. In general, skill in the use of tables is much needed. I would favor even more drill on use of tables, bringing in, perhaps, different forms of tables unfamiliar to the student, in order to accustom him to ready use of new tabular forms. Otherwise, the skills needed are few, and are usually overemphasized in drills. Skill in analytic trigonometry is seldom needed, and is usually totally absent on the few occasions when it is needed. Skill in the drawing of trigonometric graphs is needed;

curiously enough, this skill, really needed, is not commonly established by those who drill to excess on many other things.

QUESTIONS REGARDING CONTENT. REASONS FOR AND AGAINST

The question of content should be based upon the aims of instruction. As Professor Spaulding has pointed out, a claim for general "mental training" is not a sufficient reason for selection of a topic for inclusion in such a course. Again, as he has stated, the fact that a given topic or a given proof is difficult has no relation whatever to its value.

Definitions. Quadrantal Values. We need not discuss the desirability of definitions, and of the relations of values of the functions in the several quadrants. We note them for completeness, and we shall discuss their treatment later.

Solution of Triangles. Tables. Logarithms. Among other topics of which there is no doubt are the solution of right and oblique triangles, with the usual laws; and trigonometric and logarithmic tables, with their uses. Again I postpone discussion of methods to be used.

Analytic Trigonometry. The first real question concerns the formulas ordinarily grouped under the heading "analytic trigonometry." Some of these are needed very seriously in later work, notably the addition theorems and the half-angle and double-angle formulas. To increase this list very much, however, is dangerous. It is my experience that students very commonly forget them all by the time the calculus is reached, particularly if they have learned a large number of formulas. Extensive work and long drill on exercises at this point seems to defeat its own purpose in that the large number of formulas learned tend to become confused in students' minds, and hence all are forgotten. The tendency to dwell upon this work arose under the "disciplinary" school of education, since such work can be extended indefinitely. If we are to follow the advice of modern educational theorists, we must choose for inclusion those whose usefulness can be made apparent, or at most those which we ourselves know to be necessary. If we do this, the list will be short, and will include little beyond what I have just mentioned. In fact, even for these it is hard to find motivation. Later I shall give some indications of motivation for those I have named.

Graphical Drawing. Functional Variation. The situation is precisely reversed regarding the topics of graphical drawing and functional variation. These topics were not highly regarded under the "disciplinary" theory, since analytic trigonometry loomed large under that theory. Under the aims now emphasized, however, none is more prominent than is the study of quantitative relations,

and the formation of habits of functional thinking. For engineers, and for practically all others, the graphical pictures of the functions are of real import; and it is curiously true that they are not readily forgotten. It is also important for engineers (and others) to form the habits of functional thought which occur here: not only graphical representation, but also maxima and minima, periodicity, slow and rapid rates of change, composition of graphs, and those other graphical changes that I may describe briefly to you by the geometric words expansion, contraction, translation. Such ideas rather than memorized formulas, are capable of "transfer" to other fields. They are readily explained to students in terms of such important concepts as wave motion, sound, and so on. In this field, then, I believe we should go much further than we now do. The plotting of such forms as

$$\sin x + \cos x, \sin 2x, \cos (x + a), \sin x + (\tfrac{1}{2}) \sin (2x),$$

and quite a few others of about equal difficulty, can be made real, interesting, and permanently valuable. Finally, such work gives a real motivation for the analytic trigonometry which I had mentioned above. In all, it appears to me that we have not done enough of such work.

Other Topics. There remain an assortment of other topics of a minor kind, some of which have been emphasized unduly into rather major topics. Some of these I believe we must retain; others I feel are unsuited to a beginners' course, and should be absolutely omitted here.

Doubtless we should retain a brief treatment of radian measure. It is important later, particularly in the calculus, but the motivation for it lies precisely in the calculus, where we wish to avoid an inconvenient multiplier in the formula for differentiating $\sin x$, etc. This is extremely hard to explain at this point, and the motives that we can give are likely to be a bit hypocritical. Even in graphical drawing, radian measure is by no means necessary. However, it is at least one legitimate motive which we can explain at this point, if, indeed, the graphical work has not been slighted. At most, however, radian measure should be treated very briefly.

Inverse functions should also remain, but again the real motivation lies in the calculus, far beyond the present stage. The real importance of the inverse functions certainly rests upon the fact that their derivatives are algebraic, so that the integrals of rather simple algebraic functions happen to be these inverse trigonometric functions. Otherwise, the motives at this point are very meagre. Graphical drawing again comes nearest to furnishing a true motive. Another, which we do not commonly use, consists in the question: Given $y = \sin x$, find x if y is given.

Trigonometric equations are sometimes stressed almost to the point of becoming a major topic. I see no justification whatever for this. Such equations do not even occur in higher branches of mathematics to any considerable extent, let alone in those branches of pure and applied mathematics which the student is likely to take. We can justify a few very simple ones. Thus I have just stated the equation $y = \sin x$, whose solutions are really important. But those commonly given are trick examples of much more complicated form. I would advise that equations be kept very simple indeed, going not much beyond $\sin x = \cos x$; and I should advise that the graphical solution, even for the one just stated, should be emphasized along with the algebraic solution.

Vectors may be mentioned. They are real to the student through forces. They may be used conveniently as illustrations, and they form a motive for some work, as I shall point out.

De Moivre's Theorem, however, and imaginary quantities, though they are suggested to the adult mind at this point by the mention of vectors, seem to me to have no place in an elementary course. I would strongly advise their total omission here. Otherwise I should want to know some justification in terms of modern psychology; I myself know none.

Infinite series also seems to me to have no place in such a course, though it has sometimes been given a large role. It is eminently true, of course, that infinite trigonometric series are of tremendous importance, but the reasons for this lie far beyond the students' present grasp. Rather I would spend the time in preparing his mind for the real importance of such series, and this can best be done by showing here the graphical composition of functions which I have mentioned and emphasized already.

Spherical Trigonometry. Whether there is to be any work on Spherical Trigonometry or not is in most schools a question of time. In any case, however, the fundamental laws can all be stated in very compact form. Even for such profound purposes as celestial astronomy, my friend, Professor F. H. Seares, Assistant Director of the Mount Wilson Observatory, prefers to use, and to have those under him use, formulas that can all be both derived and written out in full on six pages. Elaborate work on spherical trigonometry is therefore not needed, even for extreme cases, and the attention of the student may well be held to a true grasp and real comprehension of the fundamental facts, which are really few and simple.

Questions of Arrangement

I have not mentioned above any questions of arrangement in an order, since the question of selection of topics is quite separate from their arrangement.

Definitions. Ratios in Right-Triangles. General Values. Line-Values. It is obvious that the definitions should be given first, or at least just after a statement of the various geometric possibilities in the solution of triangles. But the matter is not quite so simple as that. Shall we give first the definitions for right-triangles, or shall we give at once the general definitions? Shall definitions in terms of the so-called line-values be given?

If students were wholly unacquainted with the trigonometric ratios, I would incline toward the right-triangle definitions first. However, as some of you may not know, strong forces exist that will shortly compel the teaching of the first definitions as a part of elementary algebra in every high-school. This has been strongly recommended, on account of the functional concepts involved, by the National Committee, and it has been adopted as a definite and compulsory part of the definition of elementary algebra by the College Entrance Examination Board. Without explaining further their reasons, I would point out that we may assume that all students have at least an elementary concept of *sin*, *cos*, *tan*, as defined for right triangles.

For this reason, and because the practice of giving two sets of definitions has been a source of confusion to students, I would advise giving the general definitions from the start, with, of course, an explanation of their agreement with the right-triangle definitions in that case. This will simplify our problem, and I see no serious dangers in it.

Definitions in terms of the so-called line-values have become unpopular, and I know that there are real and serious objections to them. However, the idea is very useful, and I feel that we have lost something by total omission of it. Naturally it would be confusing to introduce these early, until the student has fully digested the general definition. One suitable place would be in connection with one of their uses, namely, in connection with graphical drawing. This can be done very simply and quickly, and without calling it a new definition, simply by drawing a circle of radius 1, and using it for geometrically constructing graphical curves.

Solution by Natural Values; by Logarithms. Another question of arrangement has to do with the introduction of logarithms, and the solution of triangles by logarithmic means.

The present development of computing machines has changed materially the effectiveness of natural values. Many computers have said to me that five-place tables are no longer useful for logarithmic computation, and it was said by one speaker at the recent Napier celebration that logarithmic tables must be over seven places in order to be as effective for professional computing as are the modern machines with suitable natural values. I myself have

been approached three times within the year, each time by an engineer in a large engineering corporation, concerning the publication of seven-place natural values which the writer wished to prepare and publish. While I have been unable to secure commercial publication of such a table, there is no doubt that many engineers wish to use it, of course with computing machines.

While our students cannot have access to computing machines in order to do their problems, it is therefore not proper to tell them that logarithmic solutions are to be preferred in all situations. In the field, of course, logarithms will continue to hold their own to some extent, on account of the difficulty of reaching a computing machine; but in the field, less accurate work may suffice, and the slide-rule may be used; it is, in effect, a small computing machine.

I conclude that we should at least teach students the solutions of triangles by natural values first, including even the use of the law of cosines, which is not difficult if a computing machine is available, or if the student has, as he should have a good five-place table of squares.

The logarithmic solutions should not be omitted by any means, of course, but they may be separated distinctly from the solution by natural values. This makes possible a distinct break at logarithmic solution, and makes advisable the postponement of the introduction of logarithms until after the entire solution of triangles has been handled by natural values. The law of tangents, to which I am about to come, will be put over with logarithmic solutions, of course.

Place of Law of Tangents. Half-Angle Formulas. The law of tangents, and the associated half-angle formulas, were usually given with the work on analytic trigonometry in the traditional books in which analytic trigonometry was much emphasized. This resulted in the postponment of the logarithmic solution of oblique triangles until after analytic trigonometry, and it was often stated as a justification for the lengthy work on that topic. Several very simple geometric proofs of the law of tangents exist, however, and many of the present textbooks on trigonometry give such a proof. It is no harder than the proof of the law of cosines, and is certainly not so hard as the proof of the addition theorems, which were a necessary group of lemmas in the old order. It is certainly unjust to the student and to the subject to postpone the proof on the plea that we must give some motive for the analytic trigonometry. Such a motive is no motive at all, but only a half-veiled hypocrisy.

The legitimate place for the law of tangents, and the associated laws, seems then to be with the chapter on logarithmic solution of triangles, separated, as I have just remarked, from the solution by natural values.

Place of the Addition Theorems and Derived Results. It is a

corollary of the preceding paragraph that the work on analytic trigonometry should follow rather than precede the work on logarithmic solution of triangles. This has the additional advantage of grouping all of the work on solution of triangles together, rather than with a gap filled by matter of different type. I have already advised that this whole work be limited to the addition theorems and results that follow immediately from them.

Place of Graphs and Composition of Graphs. Graphs may well occur before analytic trigonometry. My present paragraph follows, in order to be a unit, for some of the work on graphs should follow the analytic work; and it is that which I wish to emphasize. An early chapter on graphs would be limited to the graphs of the functions as defined. Such graphs are very simple, and an early introduction of them may be made as an incidental part of an early chapter. The more serious graphical work, which I have already discussed, must follow analytic trigonometry, because it should use the addition formulas and the multiple-angle formulas. It will also form a legitimate motivation for the preceding work on analytic trigonometry, as I have remarked.

Methods of Teaching

Approach to Crucial Topics. There are several points at which the approach must be carefully made in order to avoid destruction of interest. The student should never be placed in the mental attitude which leads him to ask "Why do we have to study all this stuff?" If that question is asked, the teacher has committed an error. It should be forestalled. The solution of triangles is perhaps an exception, since the motive is obvious. Even here, however, a brief treatment of the geometric determination of triangles by given parts is by no means amiss.

Motivation. I have said that the motive for the solution of triangles was obvious. Such is not the case for many another topic. For the addition theorem, for example, no motive is apparent in most books. A correct motivation here can be had at once through the laws for forces (or vectors). If a force be projected first onto a line that is not horizontal, and thence onto the horizontal, the result will be a part of the projection of the original on the horizontal. The other part is the projection on the horizontal of the second component in the original projection. What I have here stated is, in words, the addition theorem for cosines. I had mentioned before certain graphical work as a legitimate motivation for the same law; but it cannot be used at first because the graphical work must wait for the law.

For graphical work itself, motivations are also needed. This is

furnished in full measure by reference to wave motion, sound, and so on, if the teacher will mention these things. As you have heard from Professor Spaulding, it is highly advisable that the teacher should mention them, even at the cost of some time and perhaps of a little effort on his own part. To omit these thing on the plea that there is not time, is to return to unmotivated work, to work without interest, to drudgery that has as its reward dullness and resentment and forgetfulness.

For some topics, as I have said above, motivations are difficult to find. In those cases I have advised either total omission (as in the case of De Moivre's Theorem and Infinite Series) or else suppression to brief mention (as in the case of radian measure and inverse functions and trigonometric equations). For some of the latter, I have been able to find a few half-motivations. Thus the student can be told (and shown) that it is usual to draw graphs of trigonometric functions in radian measure. Another motivation for radian measure can be found in the relative simplicity of the calculations of linear speed of a rotating body; and that is a good thing to insert, after the manner of the advice given us by Professor Spaulding. For inverse functions, there is only the mild motivation by means of the very simple equations which I have already quoted, and by graphs. These also simultaneously form some justification for very simple trigonometric equations. Hence, on the whole, if the work on these topics be kept small, I have provided some little motivation.

Attitude toward Memory. Proof as a Mnemonic. As my own experience goes, many teachers and most students have a faith in memorization that is entirely unwarranted. Normal people cannot remember long lists of formulas, and the teaching on that assumption is entirely wasted. What normal human beings have to do is to select a very few formulas that they will really memorize thoroughly, and then derive the rest from them. If the student can be shown that proofs can often replace memories, his attitude toward proof may change. Personally, I advise rote memory only of the addition laws for sines and cosines; and I personally always derive every other formula from these every time I use them; of course I can do this rather quickly. But nobody need accept my own plan. What everyone should do is to formulate a plan of his own, which fits his particular mentality; the poorest plan of all is to decide to remember everything. That is like the housekeeper who decides to keep all old articles in the attic against some possible future use: the result is that the attic becomes a wild mess in which even the few things that are of real value cannot be found. Far better to rid the attic of all but a few possessions whose future usefulness can be predicted certainly, and to arrange those few in

careful, orderly fashion, so that they may be found quickly when they are needed. The human brain is like that. I have helped many a student to clean out his mental attic of trash collected there on the advice of junk-minded teachers. To them I say, as I say not to you, that a memory which is not to be a permanent memory is a waste of time.

Transfer of Ideas. Projection. Forces, Composition. I have commented above, and Professor Spaulding has commented, upon the difficulties of transfer of ideas from one field to another field. We have advised strongly that every effort be made to help the student to see possible transfers. How hard they are to see, I know, for I have had serious trouble myself. Even in the same subject, transfers frequently fail, even in teachers' minds. I know many high-school teachers (and I think there are some university teachers) who never realized that the formula for the sum of a geometric progression is *identical* with the formula for factoring the difference of two n-th powers, though they themselves have taught both formulas for many years. There are teachers of trigonometry who know well both the addition law for cosines and the rules for composition of forces, but who never happened to think of the statement I made above about the addition law in question. Certainly many teachers know how to find the resultant of any two forces by projecting those forces onto the two axes; and they also know how to find the third side of a triangle if two sides are given in length and in position; without, however, ever having seen that the problems are identical, and that we solve them habitually by different processes only on account of our inertia. That two simple harmonic motions can be combined into one means that such forms as $A \sin x + B \cos x$ can be written in the form $k \sin (x + y)$ where $A = k \cos y$ and $B = k \sin y$. The graphical solution of these equations is very easy. It happens that all these topics are of quite a little importance, but I know (by my own experience) that people often know everything that is necessary and yet fail to put together the elements. Such transfers occur within the same subject; if they are hard, how much harder are transfers to a wholly new field! Through such as these, we may accustom students to the making of transfers, and we may hope to save them when they face new ones.

Emphasis. Relative Importance. Associated with the question of memorizations and with that of transfer, are the questions of relative importance of topics or of formulas, and the decisions regarding emphasis to be placed upon each one. It is idle to say that all are important; that all should be emphasized. I have pointed out that the human mind does not work that way. It is the teacher's business to make such decisions, and to lay the emphasis where it belongs.

This importance (or value, if you like) is not at all dependent on the difficulty of the thing, as both Professor Spaulding and I have said. Importance (or value) depends upon the possible applications, within and without mathematics; upon the usefulness of the thing studied. To know this, one must think of possible transfers, to know what a given formula or fact may imply in other fields. One must think also of the value of a given thing as a key fact or key formula upon which others may be made to depend. As an example, I have recommended the addition formulas for sines and cosines; I would place the double-angle formulas in a totally different and very inferior category.

Notation. Nomenclature. Lettering. Such minor details as notation, nomenclature, and lettering deserve quick mention. Students should be familiar with standard forms in all of these. A very excellent report was made to the American Engineering Standards Committee by a Subcommittee of which Professor Huntington was chairman; it has been adopted first by all of the major engineering societies, and later by both of the national mathematical societies. I regret that textbooks and teachers have not followed it carefully, and I would advise that they do so. To mention only one instance, the official recommendation is that ctn be used as an abbreviation for cotangent; since cot is likely to be confused with cos in handwriting.

Drawings. Sketches. Accurate Figures. Orientation. Drawings should be relied upon as checks against computations far more than we commonly do. In all, the habit of checking should be insisted upon throughout mathematics; and this particular instance of it is effective, easy, and illuminating. Sketches often suffice. Accurate figures should be insisted upon from time to time, at the least. The orientation of figures should be borne in mind. I know that many students are completely baffled by figures that appear in a position not usual to them. Essentially this is again a case of the difficulty of transfer, and it needs attention as does every case of transfer.

Conclusion

I have dwelt, perhaps too long, on what is after all one of our simplest subjects. It is my hope that my conclusions will seem to you to be sound, and also that the various principles upon which I have discoursed from time to time will seem to you to have a wider application than to this one subject alone. Indeed, I hope that they will color much that follows in this Session.

SANITARY ENGINEERING EDUCATION AND RESEARCH AT THE UNIVERSITY OF NORTH CAROLINA

By THORNDIKE SAVILLE

Professor of Hydraulic and Sanitary Engineering, University of North Carolina. Chief Engineer, N. C. Department of Conservation and Development

The sanitary engineer is not an individualist. Ordinarily he is employed by a state, a municipality, or the federal government to devise and enforce ways and means of protecting the health and promoting the comfort and convenience of the public as a group. Unlike the doctor or the lawyer, individualists whose professional relations are usually with single persons and for specific emergencies or difficulties, whose fees and standing are dependent chiefly upon their own reputation and the wealth of their clients, the sanitary engineer is commonly a whole time employee of a public agency, on a very moderate salary, whose responsibilities for the health of entire communities go on ceaselessly.

That highly specialized knowledge which, with due regard to economy, designs, constructs and operates water supply and purification works, sewerage systems and treatment plants for sewage and industrial wastes, often costing millions of dollars, brings no special gratitude to its possessor from the public which it serves. Unlike the surgeon who performs some delicate operation, or the lawyer who saves a great sum of money for his client, the sanitary engineer must usually accept as his reward a satisfaction in so successfully applying the great forces of engineering and sanitary science to the protection and convenience of the public as to make his existence virtually unknown. It is when there is a serious typhoid epidemic, a bad taste or odor in the water, an outbreak of disease due to contaminated oysters, a nuisance created by the discharge of liquid wastes into streams, or a thousand and one other interferences with the high standard of health and convenience which we now demand, that the public begins to be concerned about the sanitary engineer.

In the rapid urbanization and industrialization of the South, provision of adequate and safe water supply and satisfactory disposal of sewage and trade wastes are major problems. Thus a single water supply project may involve studies of rainfall; stream flow; population growth; reservoir capacity; dam design and construction; pipe line; tunnels; and filtration plant with coagu-

lation, chlorination, and aeration. The design of a single sewage treatment plant involves not only engineering but economic problems of a high order, to determine relative values of extent and character of treatment required as related to separate treatment by industries of their wastes, dilution afforded by discharge into a stream, and pollution of this stream by sewage and wastes from above. A single large hydro-electric reservoir may produce a potential malaria control problem involving studies of water level fluctuation, types of mosquitos, and mosquito control measures through oiling, paris green, dusting, fish, etc.

As civilization becomes more complex, problems of the sanitary engineer require an increasing scope of service to maintain an environment healthful, comfortable, convenient, and not too expensive. Sanitary engineering is the application of the sanitary sciences to engineering problems. The more important of these sciences are biology, including bacteriology, entomology and microscopy; chemistry, especially bio-chemistry and colloid chemistry; epidemiology, particularly as relating to diseases carried by water, food and insects; sociology; and last but not least, hydraulics, for almost every problem in sanitation involves a thorough knowledge of the principles governing the origin, flow, storage, circulation, and control of liquids and gases. Sanitary Engineering is the correlation of these sciences and their application by the sanitary engineer to control environment and prevent disease.

Sanitary Engineering Education

It may be seen that adequate instruction in sanitary engineering must not only be highly specialized, but that it requires teaching personnel and laboratory equipment of a high order. The demand for trained sanitary engineers is restricted largely to federal, state and municipal service. This demand is constantly growing, especially in the South, and may be met by two classes of engineering graduates:

(*a*) Graduates of engineering schools, with a four year course, who have had, in addition to a broad fundamental training in civil engineering, one or more special courses in sanitary engineering. Such graduates have a background fitting them first to perform routine duties such as filter plant operator or inspector, and later to work up to general administrative positions such as city engineer or city manager, where a general knowledge of sanitary engineering is, or should be, a requisite.

(*b*) Engineers who have completed one or more years of post graduate study in an engineering school having personnel and equipment available to give a highly specialized training in advanced

sanitary engineering and allied sciences. Such graduates are fitted to become experts in special fields, such as the design or operation of water or sewage treatment plants, municipal refuse disposal, water supply design, shellfish or milk sanitation, etc.

The last survey of courses in sanitary engineering in the United States was made by I. W. Mendelsohn of the U. S. Public Health Service in 1929.* This list included chiefly those institutions prepared to give training in sanitary engineering under *both* items (*a*) and (*b*) above. There were nineteen institutions named, of which three were in the South; the Universities of North Carolina, Texas and West Virginia.

The University of North Carolina was among the first educational institutions in the South to develop in its Engineering School specialized instruction in sanitary engineering. The first undergraduate and graduate courses in this subject were introduced in 1920 and 1921, a specialist in sanitary engineering having been added to the faculty in 1919. Because of this early initiative in the field, and because of the persistent desire of the administrative authorities to develop this branch of engineering, the sanitary engineering division of the Engineering School of the University has continually expanded until it is at present one of the best equipped in personnel and laboratory facilities in the country.

Curriculum in Sanitary Engineering

The development of specialized training in sanitary engineering at the University of North Carolina has followed from its inception certain well defined guiding principles which are in some respects unique in such instruction. Many of the individual principles in engineering education recommended by the Investigation of Engineering Education † as having been found to be successful at various institutions, have long been combined as a part of the curriculum at this institution. These guiding principles may be enumerated as:

1. A fundamental four year civil engineering training.
2. No particular emphasis upon strictly sanitary engineering subjects in the undergraduate years.
3. An unusually complete and well rounded course in general sanitary engineering problems in the senior year.
4. The coöperative system in the junior year.
5. One or more post-graduate years with intensive and specialized courses in sanitary engineering subjects.

* Sanitary Engineering Courses in Engineering Colleges of the United States, by Isador W. Mendelsohn, U. S. Public Health Service Reports, March 22, 1929.

† Report on Investigation of Engineering Education, 1923 to 1929, by the Society for the Promotion of Engineering Education assisted by the Carnegie Foundation, Engineering Foundation, and Industries.

6. Small classes and well equipped laboratories.
7. Instructors who are experts in their respective fields, and whose academic duties are allocated almost exclusively to teaching and research in sanitary engineering.
8. Encouragement of research in sanitary engineering fields.
9. Encouragement of instructors to associate themselves in the summer with state and municipal engineering agencies concerned with sanitary engineering problems, and to engage in the practice of their specialties.

Discussing briefly the application of these principles, it may suffice for the first two points to say that a thorough training in the fundamental subjects of civil engineering (of which sanitary engineering is a branch) is regarded as indispensable. Unlike certain other institutions specializing in sanitary engineering education, instruction in such subjects as bacteriology, limnology, and biochemistry, is confined to the graduate years. A principle has been followed to develop in the first three years a well rounded background in the fundamentals of civil engineering and in cultural subjects, particularly English, which would be impossible if specialization was attempted before the senior year.

CORNER OF LABORATORY OF SANITARY BIOLOGY.

In the senior year students take a rather complete course covering the entire field of sanitary engineering. The course is

given by two instructors, each a specialist in different fields. A brief consideration of the history of public health is given, including the researches of Pasteur, Koch, Ross, and others and an epidemiological background is presented, stressing particularly the engineering aspects of control of typhoid fever, dysentery, cholera, malaria, yellow fever, and other diseases controlled by sanitary engineering methods. Following this a rather extensive treatment of water supply is undertaken, including the actual design of a water supply project involving studies of rainfall; stream flow; storage; population growth; economic design of pipe line; power and pumping installation; dam construction; etc. Design and construction of sewerage and drainage systems is also considered, both in theory and in an actual design problem. Under another instructor the principles of design and operation governing treatment plants for water, sewage, and industrial wastes are taken up with the theory and practice of laboratory control of such processes and their relation to stream pollution. Students are required to prosecute the laboratory control of the University water purification plant, which is adjacent to the sanitary engineering laboratory, and to make routine laboratory tests for sewage plant operation and milk control.

In the junior year the Engineering School has adopted the coöperative system in Engineering Education, whereby half the class is alternately employed for three month intervals on some engineering work. Students interested in Sanitary Engineering usually are placed in water or sewage treatment plants where much valuable practical experience is gained which makes later theoretical instruction of added significance.

With a well-rounded background, the student ambitious to qualify as an expert or interested in research, is now fitted to undertake one or more years of post-graduate work. Advanced courses are offered in such subjects as Hydrology, Sanitary Chemistry and Biology, City Planning, and advanced Sanitary Engineering. The latter includes special consideration of design and operation of water and sewerage systems and treatment plants, and other sanitary engineering works.

Classes are deliberately restricted to not over twenty in the senior year, and not over five or six in the post-graduate courses, to effect both a selection of students and close personal contact of students with the instructors.

Probably in no branch of civil engineering has so much progress been wrought through modern laboratory and plant research as in sanitary engineering. New developments in theory and practice are the order of the day. The facilities for both plant and laboratory instruction and research are unique at the University of North Carolina. Views of the sanitary engineering laboratory are shown

in the accompanying illustrations. These are fully equipped not only for routine bacteriological, biological and chemical work, but have adequate facilities for prosecuting special research problems. In addition, the water purification plant of Chapel Hill is connected directly with the laboratories, and both it and the sewage treatment plant are constantly used in connection with instruction and research.

Professors and graduate students have in recent years made significant contributions to the advancement of sanitary engineering science through investigations conducted in these laboratories and plants.

CORNER OF LABORATORY OF SANITARY CHEMISTRY.

As evidence of effective coöperation with private and public agencies in the investigation of scientific problems related to hydraulic and sanitary engineering, there may be cited the following grants to the Engineering School for research in these fields:

(*a*) From the city of Durham, N. C., about $4500 for investigations of textile wastes and the possibility of treating such wastes in combination with domestic sewage.

(*b*) From the Chlorine Institute, $2000 for the study of effects of chlorine on the reduction in the biochemical oxygen demand of sewage; on sewage treatment plant efficiency; on elimination of odors; and on improvement in quality of streams receiving sewage.

(*c*) Chemical Foundation, $1000 for studies on treatment of textile wastes to render them suitable for discharge into streams or sewerage systems tributary to sewage treatment plants.

(*d*) Municipalities State and Federal Agencies about $4000 during past three years, for various studies relating to sewage treatment, stream flow, evaporation and silting, filter and beach sand characteristics, rainfall and run-off relations, and miscellaneous hydrological investigations.

Committed to a correlation between theory and practice, the students in sanitary engineering are further advantaged by the fact that one of the professors in sanitary engineering is chief engineer of the State Department of Conservation and Development, and another is a principal assistant engineer of the State Board of Health. In this way students are constantly brought in touch with actual problems engaging the attention of state agencies, and post-graduates are frequently engaged upon specific research problems of one or the other of those agencies.

The acquirement of specially trained personnel and development of the special laboratory and other facilities for instruction in sanitary engineering is probably warranted at only a relatively few institutions. For this reason the Engineering School at the University of North Carolina attracts many graduates of other institutions who desire post-graduate work in this field. Recent graduate students have come from as far west as Oregon and as far south as Venezuela. Graduates of the Engineering School have accepted positions with such federal agencies as the U. S. Geological Survey and U. S. Public Health Service, with various state boards of health, and with many sanitary engineering departments of municipalities in different parts of the South.

Throughout the past ten years the specialized work in sanitary engineering at the University of North Carolina has been developed with the ultimate objective of furnishing first a solid background in this science for its own undergraduates, and second, in providing those highly specialized and expensive facilities for graduate and research work which would prevent inefficiency through needless duplication elsewhere in the vicinity, and would be attractive to advanced students from other institutions, not only from the associations and advantages of location at a great southern educational institution, but because of the unusual opportunities offered for contact and practice in the sanitary engineering problems of the South through the unusual affiliations with the sanitary engineering work of several state agencies.

COLLEGE TRAINING OF ENGINEERING STUDENTS FOR WORK IN MACHINE DESIGN *

By JAMES A. HALL
Brown University

In any attempt to list the topics which should be included in the technical training of men who may enter the general field of machine design, it seems desirable to consider the mental equipment expected of a first class designer in responsible charge of such work in an industrial establishment. The question then can be raised as to how extensively and in what ways the technical school should attempt to furnish this training and what part of it should be left for the prospective designer to secure from practical experience or individual study after graduation.

One important requirement of a designer is knowledge of how the parts of a machine or mechanism can be made and what factors are involved in attaining maximum economy of manufacture. This presupposes a general knowledge of the problems of casting and forging the common engineering materials and familiarity with the fundamental machine tools. However, pressed metal and welded construction have become of major importance in the replacement of castings. Furthermore the parts being designed probably are to be made in large quantities and this may involve jigs or special tooling on manufacturing type machines, or the use of special purpose equipment. New methods of manufacture are being developed and these must not be overlooked by the designer as a change to conform with them may result in a considerable saving in cost.

This raises the question of the extent to which methods of manufacture and particularly the technique of mass production should be included in the college curriculum. The average shop course gives some practical training in making individual pieces and very little in the methods required in manufacturing large quantities. Furthermore, it is practically impossible to maintain up-to-date machinery for the former and out of the question, from an economic standpoint, to have equipment covering the latter. Emphasis may be placed on the problems of large scale production by discussions, required reading and inspection trips, and through this the importance of the subject may be brought home to the

* Presented at annual meeting of the S. P. E. E. at Purdue University, June 17 to 19, 1931.

student and his interest aroused in securing additional information for himself.

The problems involved in combining economically the parts into the finished machine are also of great importance. The designer should make his layouts in such a way that sub-assemblies are used as far as possible, and that the final assembly consists primarily of fastening such units onto the main frame or casting. In this way the savings due to working on smaller units are realized. Any fitting of parts on the assembly floor is expensive while accurate alignments and close adjustments add greatly to the time required. A careful study should be made of each design with due regard to the effect of the different elements on assembly cost. Recently in the redesign of a line of textile machines, such a study reduced the total cost of assembling to one-third of the former value.

The designer is frequently concerned not with a single machine but with a complete line, and here the question of standardization becomes a vital factor in economy. In a case recently seen by the writer one type of end frame had been made standard for fifteen machines, replacing the fifteen different ones formerly used. Any reduction in the variety of parts to be made and stocked together with the resulting increase in the lot sizes makes for an improvement in manufacturing methods, lower shop costs and a saving in inventory of parts and of goods in process. Even when special machinery is being built, the development of standard parts and standard mechanisms is an economy which must be considered by the designer. He should also be familiar with national standards on screws, keys, tapers, shafts, gears, etc.

Among other factors which the designer must take into account is the accessibility of parts which may require adjustment or repair. The convenience of the operator is of particular importance if the new design is a machine to be used in manufacture. This includes such items as location of controls, height of work, appropriate speeds and feeds and facilities for making rapid changes.

Engineers are usually noted for emphasizing utility and neglecting aesthetic values, yet it is generally recognized today that appearance has an influence on sales even of factory equipment. Why shouldn't the workman who must spend a large portion of his conscious life in a factory be surrounded by good looking, as well as properly functioning machinery? The first class designer, somewhere in his experience, should acquire a sense of proportion and appreciation of good lines which will be reflected in his work.

Up to this point consideration has been given to factors in the mental equipment of the machine designer apart from these all

important subjects: force and velocity analysis, the design of machine elements and engineering materials; all of which receive a certain degree of emphasis in mechanical engineering curricula. Before going on with the latter, the question may well be raised as to where the designer is to acquire these other qualities essential to his success.

One solution is to depend on industry to supply any training which the student is to get along this line. The college then furnishes theoretical background and training in analysis, these being the subjects it is best fitted to give and which the student will find most difficult to acquire by individual study. After graduation he is expected to enter industry as an apprentice and spend his first two or three years learning manufacturing methods and the technique of design, partly in training courses and partly by working under men who have already qualified as designers. This has the added advantage that information peculiar to the particular type of industry is acquired at the same time as the general training. A disadvantage of this scheme is that many of the smaller plants which need engineers with the analytical background of college training do not have properly qualified men to give the practical guidance required, and the engineering graduates going with such concerns must struggle on with such mental equipment as they may have.

In some engineering schools the students are required to carry through more or less complete designs of a number of machines. This gives the opportunity to discuss methods of manufacture and assembly and similar practical problems. An effort has even been made to teach some ideas of standardization between machines by correlating the designs of a number of students, each of whom is working on a different size.

This method is open to two general objections. The teacher as a rule is not an expert in all the fields of design involved and probably uses the same problems year after year. In the engineering offices of a company manufacturing such machines, changes are continually being made either to improve operation or to take advantage of new methods which reduce costs. The engineers in the latter think of design as something which is alive, continually improving and developing, while the student, on the basis of his problems, may consider it a routine, standardized proposition.

Another disadvantage of carrying through complete designs in school is the time required. For any such design, either in school or in an industrial establishment, there is a large amount of routine work compared with the time required for solving problems which really challenge the intellect and hold the interest of the student. Furthermore, the designer may be spending forty-four hours a week

on the job while the student has but six or eight for this part of his work. With the increasing volume of analytical material which should be brought into the design courses, the writer feels that there is not time in the four-year undergraduate curriculum to lay out complete designs of machines, even if this practice had desirable features from an educational standpoint.

A third alternative between the purely analytical course which does little with the practical features of design noted earlier, and the type in which complete layouts of machines are made, may be found in the development of the case system. In the early stages of his practical experience the machine designer rarely is called upon to design a new machine from the ground up, but he is primarily asked to study improvements in machines already being built. These may call for changes in design to make use of some new method of manufacture, to facilitate certain types of adjustment or repair, to make possible certain unit assemblies, to standardize certain parts of a group of machines, to introduce some new feature, and so forth.

Practically every type of problem, both practical and analytical, can be illustrated through such cases. A great deal is learned from the study of the related parts of the machine, and the reasons for the change. It also stimulates the proper attitude toward design as a growing subject always calling for new ideas. As far as the writer knows, the method has had little use in machine design instruction, but he feels that it has possibilities in focussing the student's attention on selected practical principles of design, and that it is both more effective and requires less of the student's time than the method of laying out complete machines.

While the requirements already taken up are of great importance to the machine designer, he can readily learn them through practical experience in a good organization. On the other hand it will be only with the greatest difficulty that he can acquire the ability to analyze complex problems in terms of forces, velocities, internal stresses, deflections, etc., unless he has already acquired a good theoretical background for this work. The main emphasis of the college course in machine design should be, therefore, on the application of the principles of mechanics to practical problems.

In his own undergraduate work the writer remembers a course called kinematics, a considerable part of which preceded the formal work in engineering mechanics. This course was devoted to the study of various mechanisms and to velocity analyses. A little attention was paid to forces, and to acceleration, but friction calculations and the determination of inertia forces in translating and oscillating bodies were entirely omitted. Of course, these subjects were later treated in the mechanics course and some illustrations

came into the formal work in machine design, but a valuable opportunity was lost, both for training in methods of analysis and for making a real challenge to the intellectual capacity of the student.

The practice now followed in many institutions of teaching the engineering mechanics early in course and following it by the study of mechanism, probably called mechanics of machinery, is a great improvement. It is then possible to analyze completely the mechanisms studied, not only from the standpoint of velocity but also of static, inertia and friction forces, and in some cases taking up stresses and deflections. With modern refinements in design, friction cannot be ignored, and in this day of high speed machinery the inertia forces involved in cams and linkages are an important factor. The writer has found that this method of approach, with its test of analytical ability, makes a real appeal to his students.

This work can be tied in closely with the design of machine elements. Here two points of view are possible. One is to give as much practical information concerning design as possible so as to make the student of the greatest immediate use, while the other is to develop a critically minded attitude. An illustration may be taken from gear teeth curves. In the first case the topics taken up will include definitions, standard curves, standard systems, methods of drawing tooth and manufacturing processes. On the other hand the teacher may point out that in these days of mass production special gears are economical if they are better suited to a particular purpose. Emphasis is placed, therefore, on the principles of conjugate curves and of tooth action. The factors involved in using gears with long and short addendums or special pressure angles can be analyzed, and the possibility of using standard cutters to generate special pressure angles becomes apparent. The student from such a course will not be amazed when he finds chain sprockets being cut on a spur gear hobbing machine or odd shapes being generated on a Fellows gear shaper, and he will probably have ideas of how to develop the contour of the cutters.

In the study of the strength of gear teeth the class may concentrate on the Lewis formula and get facility in its use by the solution of many problems. In the other method of approach emphasis will be placed on the fact that the Lewis formula, while very valuable and generally giving conservative results, is only an approximation. It will be pointed out that the calculated tangential pressure at the pitch line is not the actual bending force at the end of the tooth and that the Lewis factor taken from the tables is true for one value of the fillet radius while the method of generation may have produced quite a different value. Methods of determining the factor for any given case may be developed. It can be pointed

out that the Lewis formula is based on the strength of a beam, while modern gears generally fail by wearing out, and the Hertz equations for determining local stresses due to the pressure between curved surfaces may be brought into play. Then studies of gear tooth models by the use of polarized light may be introduced to show that a tooth is not loaded like a mathematically correct cantilever beam but that there are great concentrations of stress at the fillets and at the point of contact. These and other factors coming out in recent studies will give the student a more truly understanding attitude toward the problems involved in gear drives.

The design of bearings is a particularly interesting example of the two methods of approach. On the one hand practical information concerning materials, adjustment and methods of lubrication can be given, and tables of practice or empirical formulas can be used in choosing allowable pressures per unit area, relations between pressures and rubbing velocity and probable coefficients of friction. On the other hand the teacher may start with discussion of the differences in the friction of oily surfaces and those separated by oil films. The conditions required for maintaining such a film may be taken up and calculations introduced showing the effect of the clearance or film thickness and of the viscosity of the oil on the coefficient of friction. The point of minimum pressure may be determined and the mathematical analysis used to calculate the permissive loads before the film thickness is reduced to a dangerous point. Even here the teacher must point out the limitations of the mathematical theory in that the effects of end leakage are not taken into account and point out that the designer must still depend on experimental data for much of his information concerning both lubrication and bearing materials.

The use of ball and roller bearings, brings in many possibilities of analytical study. In general the capacity of such bearings can be taken from the data book of the manufacturer, but the determination of the forces acting on them is a problem for the designer. This not only includes ordinary force analyses but deflection may be considered in calculations to determine if the permissible slope of the shaft through the bearing is exceeded. The use of double opposed bearings to secure rigidity of the ends of a shaft brings in some interesting problems either in fixed end beams or continuous beams with more than two supports and these calculations may reveal excessive loads on the individual bearings.

In the desire to cover a large number and variety of problems in the time available, the teacher of machine design is tempted to refer the student to handbooks or textbooks for rules or equations which may possibly have a rational basis, but to many students under pressure to complete their assignments these become simply

empirical formulas. This in many ways violates the ideal of college-type instruction which should be to develop the ability to reason on the basis of thoroughly understood fundamental conceptions. During a fairly extensive practical experience the writer has met relatively few analytical problems in design in which the easiest approach was not to start with the fundamental conception in mechanics or elasticity and build up the equations or formulas needed. If the same practice is followed in teaching, the students get a real grasp of the fundamentals, ability in analysis and an appreciation of the limitations of the formulas used.

When this general method is followed the class moves more slowly and spends more time on each subject than where the object is to give the greatest facility in the use of formulas. However, the result from an educational standpoint is much more worth while. For instance, solution of problems on the critical speed of shafts basing the work on empirical equations hardly seems worth while, but a somewhat longer time spent in developing fundamental relations in a number of simple cases involving vibration may be highly valuable to some embryo leaders in machine design who have thus been given a foundation for future study.

A little earlier in the paper reference was made to the fact that machine design should be taught in such a way that the student is impressed with the fact that it is a live and developing subject. This is just as true with regard to machine elements as with manufacturing processes. Frequently the design of these elements in college courses is based on standards developed years ago from the research or the practice of that day, while new experimental and analytical studies may have radically changed many of these standards in our leading industries. Of course the college teacher cannot keep as up to date in all branches of design as an engineering specialist in his narrow field of industry. On the other hand, technical papers of high order and research committee reports on such subjects as lubrication, gears, springs, etc., are frequently published so that any teacher should have little trouble showing the student that machine design is continually changing and in interesting him in the kind of study which will be necessary if he is to keep up to date in his later work.

The backbone of the formal course in machine design will be made up of the study of mechanisms and of machine elements with emphasis on thorough analysis based on fundamental conceptions in the engineering mechanics and the theory of elasticity. Continual references will be made to the part which has been played by experimental work in modifying the theoretical analyses, in determining their practical limits and in developing constants and coefficients, and emphasis will be placed on the fact that recent or

current research may be of particular importance. The student should become familiar with the sources of such information and also impressed with the idea that similar experimental investigation must go hand in hand with his work in the future. The primary objectives of the teacher should be to give the student an understanding of fundamental principles, ability to analyze, and a critical attitude of mind.

While the applications of mechanics have been given first place in a college course in machine design, the study of the materials used in machine construction are of only slightly less importance. As long as this work consisted primarily of describing the different materials and giving their properties, it could well take a minor place in the formal design course and the student could be expected to add to this information by further reading as the occasion arose. The use of the microscope and the X-ray as a supplement to the older methods in the study of metals and alloys has added so many new ideas in this subject that a separate course, taught by a specialist, is indispensable. As a result, the student begins to think of metals as collections of crystals and studies their formation and growth. He also becomes familiar with such subjects as solid solutions, equilibrium diagrams, transformation points, conditions existing during heat treatment, the changes in these factors brought about by different alloys, etc. This procedure helps him build up a frame-work within which a great mass of miscellaneous information concerning materials can be classified. It is highly important for the designer to get this background so that he can really appreciate and understand the developments which are continually coming in this field.

In every industry the designer is likely to find need for some particular technical information apart from those items already discussed. The internal combustion engine brings in thermodynamics and flow of air. Even in industries which have always been thought of as purely mechanical, developments along new lines confront the designer. Interesting examples come from the machine tool field where within the last few years there has been a great increase in the use of hydraulic drives. This raises such problems as the relation between straight line and turbulent flow and the resulting friction losses, the effect of variations in viscosity between oils or due to temperature rise in a given oil, etc. The designer may have studied hydraulics in college but viscosity probably was never mentioned as water was the only fluid discussed, while now he is dealing with oil where viscosity is a factor which cannot be ignored.

Electricity is also becoming highly important in this industry. The use of special motors raises questions of characteristic curves,

and electric controls are bringing in problems of magnetic clutches and brakes, solenoids, relays and similar apparatus. Various types of thermionic tubes are beginning to find their way into machine design, and this makes another field with which the designer must have a rubbing-elbows acquaintance. These new demands emphasize the necessity of giving men who are expected to develop into leaders in the machine design field a fundamental background so that they may have something on which to build as the necessity arises.

The writer hopes he has left the impression that machine design and development is a worthy field of endeavor for many of the best graduates in mechanical engineering. The teacher must see that his students have an adequate training in the fundamental subjects which are the background of his course, and then develop power of analysis in terms of this background. He should keep their feet on the ground and develop a critical attitude by correlating these analyses with the best results of practice and research and by frequent assignment of cases or problems, all of which are drawn from engineering work. Finally he should arouse an interest in the elements affecting the cost of the machines being designed and thus develop in his students the economic point of view.

SOME ATTAINABLE OBJECTIVES OF COOPERATIVE EDUCATION

By CLYDE W. PARK
University of Cincinnati

The article by Professor Dugald Jackson, Jr., in the October issue of ENGINEERING EDUCATION summarizes concisely the principal advantages of cooperative training for engineering students. Professor Jackson's statement may be supplemented by an outline of objectives which were suggested a few years ago to a committee of the Association of Cooperative Colleges. The committee's inquiry was concerned with benefits which cooperative colleges sought to obtain for their students and which the college authorities had found to be not merely ideals but possible objectives. Although recognizing the student's welfare as the prime consideration, the committee also asked for a statement concerning benefits which the firm and the school might reasonably expect to derive from the cooperative arrangement.

A review of opinion representing the three points of view is given below.

The very existence of the cooperative system assumes that the student, the employing firm, and the school stand in the closest possible relationship to one another. For this reason it is not always possible to isolate the advantages as belonging to a particular one of the three. Granted, however, that the reciprocal element is always present to some degree, there are outstanding benefits which may properly be assigned to each member of the group.

ADVANTAGES TO THE STUDENT

Of the advantages to the student, those which have proved most important are the following:

1. He has the benefit of a natural method of choosing a profession, since the type of work which he should follow is determined by:

(*a*) Contact with actual industry;
(*b*) Practical tests of his own inclinations and adaptability;
(*c*) Intelligent participation in the shaping of his own preliminary training. It should be noted that this kind of academic self-determination is very different from a free elective system, since the wisdom or unwisdom of the student's preference is verified experimentally.

2. He has the opportunity of gaining a maximum of educational content from his industrial environment. In other words, he has a chance to

study applied science where science is applied. The most important results of this contact are these:

(*a*) A vitalizing of his theoretical studies through acquaintance with their practical applications;

(*b*) A stimulus toward original investigation.

3. He acquires an understanding of the human factor in industry. This important part of his training is made possible by two things:

(*a*) Direct labor experience on equal terms with other employees;

(*b*) First-hand contact with problems of labor management.

4. He gains certain disciplinary values which are inevitable results of his shop experience. Chief among these are the following:

(*a*) The habit of industry;

(*b*) A sense of responsibility;

(*c*) A feeling of self-reliance.

These three attributes, all of them essential to his future success, must of course be latent in his make-up. The nature of his shop work is such that these qualities are brought out and developed. If the student lacks them he soon drops out of the cooperative course.

5. He gains certain economic values which are likewise by-products of his outside experience. The principal benefits of this kind are the following:

(*a*) An opportunity for partial self-support;

(*b*) Enhanced earning capacity and bargaining power at graduation in proportion to his outside experience and his consequent advancement beyond the apprentice stage of engineering employment.

Advantages to the Firm

Even if the employing firm did not receive any recompense beyond a day's work for a day's wages, the existence of the cooperative plan would have economic justification. As a matter of fact, however, there are both immediate and ultimate advantages which make the firm an active partner instead of a mere source of industrial experience. Chief among these advantages are the following, which have to do respectively with personnel and research.

1. By employing cooperative students the firm is constantly and systematically infusing new blood into its organization. The morale of an entire working force is strengthened by the introduction of a few ambitious young employees who are known to be working their way up through the ranks. Experience has shown, moreover, that a considerable number of these student employees remain with the firm upon graduation and assume key positions. Thus, without the expense of maintaining a training school of its own, and without paying more than the standard cost of productive labor, the firm is able to obtain selected men, brought up in its own organization and trained by the university for future usefulness.

2. Through the cooperative relationship the firm is served by the university in many ways which would not otherwise be possible. The constant exchange of ideas resulting from the employment of students and from the visits of instructors affords opportunity for the discussion of

research problems which continually arise in industry. The student, as he becomes more advanced in his studies and more familiar with the industry is called upon to solve many of these problems. In doing so, he draws upon the resources of the university as a whole. Carrying this idea to its logical conclusion, as developments in one college, illustrate, the university becomes a center for basic research in the larger problems which affect the industry as a whole.

Advantages to the College

The college which maintains a cooperative course might well be satisfied with the larger service it is able to render to the student and to the employing industry. There are advantages within the school itself, however, that are directly traceable to the cooperative system. These benefits alone provide ample justification for the extra effort which the administration of the course involves. Among the outstanding advantages are the following:

1. Since the students are subjected to the double test of academic fitness and ability to handle outside work, there is more prompt elimination of students who are not qualified to carry on an engineering course.

2. Owing to the contact which all students have with industry, descriptive technical matter may be eliminated from the course, affording correspondingly greater emphasis on fundamental principles.

3. Because of this same contact with the applications of science to industry, the study of principles is made vital.

4. By utilizing for educational purposes the modern and complete equipment of large industrial corporations, the university not only affords better instruction but saves the expense of school shops and of illustrative models.

5. The alternating periods of shop and school respectively permit a fuller use of the university plant, since twice as many students may be accommodated as if they were all in school at the same time. In view of the present scrutiny of university budgets, this advantage is more important today than ever.

6. Through its intimate relationship with industry the university finds not only a humanizing influence, but also a perpetual stimulus to research in the basic principles of science.

COLLEGE NOTES

University of California.—With the opening of the semester in August, 1931, the College of Engineering moved into the new unit of the Engineering group of buildings, a handsome granite and steel structure. The College now comprises the departments of irrigation, civil, electrical and mechanical engineering, under the leadership of Dean Charles Derleth, Jr., C.E. The building is to house the administrative offices of the dean, the chairmen of the different departments, some of the faculty of each department, the engineering library, class rooms and the civil engineering drafting rooms.

One of the unusual features of the class room is the light buff colored rough wall finish which effectively eliminates sound vibrations and echoes. Excellent semi-direct lighting is provided throughout.

Several new faces are seen in the engineering faculty, among whom may be mentioned: Assistant Professor B. L. Robertson; Associate A. Ponomareff and R. G. Minarik; Assistants A. G. Foote, M. Asimow, G. L. Powell, D. O. Rusk and Mr. V. H. Cherry, Teaching Fellow.

Professor J. N. LeConte is spending a year in Europe, and while there is making a study of various hydraulic laboratories and power installations.

Professor T. C. McFarland has returned to his duties after a year's absence in the Bell Research Laboratory in New York.

A new Materials Testing Laboratory was also occupied at the beginning of the semester. In this are various shops and testing laboratories of the Civil Engineering Department, supplemented by special curing rooms for detailed study of the effect of moisture, temperature, etc., upon cement and concrete under conditions similar to those of concrete in large masses. Many studies relating to the Hoover Dam are under way.

Three models of the proposed suspension bridges between Oakland and San Francisco and across the Golden Gate entrance to the Bay have been made at a scale of 1 to 100. They are being studied for all possible conditions of loading, wind stresses, etc. A new 4,000,000 lb. testing machine will soon be installed.

An additional unit of the heat power laboratory, doubling its size, has greatly relieved the congestion of the last two years in that division.

Correlation of courses and experiments in heat power and hy-

draulics has resulted in more efficient results and almost entire elimination of duplicated efforts.

Columbia University.—During the summer of 1931, Dean Joseph W. Barker represented Columbia University at the Seventy-fifth Anniversary convention of the German Engineering Society (Verein Deutsche Ingenieure) in Cologne, Germany; at the History of Science and Technology Congress in London; at the International Congress on Illumination in England; at the Commission International d'Eclairage in Cambridge, England; (of which he is a member of the U. S. National Committee); at the Faraday Centenary of the Royal Institution of Great Britain and at the Centenary of the British Association for the Advancement of Science, both in London. He utilized the entire summer for visits to Technical Schools and Universities in Germany, France, Switzerland, and England, and studied the facilities and methods of instruction and research at the several schools. He received the cordial cooperation of the authorities of these schools in discussing their problems of laboratories, research, engineering connections of professors and pedagogical methods.

Professor A. W. Hixson, Professor of Chemical Engineering, attended the S. P. E. E. Chemical Engineering Summer School at the University of Michigan during the latter part of June and the first part of July. He attended as special lecturer and presented a paper on "An Experiment in Evaporation."

Professor L. Parker Siceloff, Associate Professor of Mathematics represented the University at the S. P. E. E. Summer School for Engineering Teachers, session of Mathematics at the University of Minnesota, August 24th to September 5th, 1931.

George Washington University.—In the fall of 1930 a new curriculum leading to the degree of Bachelor of Science in Engineering was introduced.

This curriculum provides a means of meeting the needs of students who do not wish to take one of the courses leading to the degrees of B.S. in Civil Engineering, B.S. in Electrical Engineering or B. S. in Mechanical Engineering and now includes the following subjects and hours:

English	6 hours
Mathematics	9 "
Physics	8 "
Chemistry	8 "
Mechanical Drawing	2 "
Descriptive Geometry	2 "
Materials of Construction	4 "
Theoretical Mechanics	8 "
Hydraulics and Materials Lab.	2 "
Hydraulics	4 "

Elements of Electricity	6	hours
Thermodynamics	6	"
Electives (from a supervised group of subjects)	27	"
Physical Education	4	"
Senior option (subject to approval as a group by the Dean's Council)	28	"
Total	124	"

To be admitted to the senior year of this course a student must have maintained an average grade of at least "C," which is the next grade higher than passing, for the first three years, and he must select subjects in or related to a definite field, subject to the approval of the Dean's Council. The first year in the George Washington Law School meets the requirements of this year for students planning to enter the field of Patent Law. Other groups are available in Physics, Chemistry, Economics, etc.

This year our curricula have been made to agree in credit-hours with the other undergraduate curricula of The University, namely 124 semester-hours.

Mr. Alfred G. Ennis has been appointed Instructor in Electrical Engineering, taking the place of Mr. Paul R. Nash who resigned to enter the Public Works Department of the United States Navy.

Mr. Chester A. Hogentogler has been appointed as Lecturer in Highway Engineering to fill the vacancy made by the resignation of Mr. Robert E. Royall.

Harvard University.—The Harvard Engineering School is offering this year a new course in the field of aerodynamics and the related subject of hydrodynamics. This course involves the study of the flow of fluids and of air with special reference to airplanes and airships. Other applications, however, will be included, such as certain features in the design of hydraulic turbines and propellers, and the vibration of electric transmission lines caused by wind. Due to its fundamental character this course is required as a part of the program of all mechanical engineers in the Engineering School, but it is being taken by students in other fields and by graduate engineers now engaged in industry. The lecturer is Dr. J. P. Den Hartog, who is head of the Dynamics Section of the Westinghouse Research Laboratories at East Pittsburgh.

This course is closely coordinated with a course given during the second half-year on the subject of mechanical vibrations, having as lecturers Mr. J. Ormondroyd of the Westinghouse Company and Mr. A. L. Kimball, Jr., of the General Electric Research Laboratories. Both of these courses are part of the offering in the department of Applied Mechanics and are intended to emphasize the importance of mathematics and physics in the analysis of problems arising in mechanical engineering and aeronautics.

Dr. Arthur E. Kennelly, Professor Emeritus of Electrical Engineering at Harvard University and the Massachusetts Institute of Technology has gone on a visit to Japan. At the invitation of the Institution of Electrical Engineers of Japan, he will deliver a group of lectures, during November, before five Japanese universities as the first visiting American professor under the Iwadare Foundation. He expects to return to Boston about the end of 1931.

Bringing the total of mechanical engineering courses at the **Michigan College of Mining and Technology** up to 26, "Internal Combustion Engines," a 23-week course conducted by Assistant Professor H. W. Risteen and intended for juniors and seniors, will be offered this winter and spring for the first time.

The work covers internal combustion engines in the field of the heavy industrial diesel, the stationary gasoline engine, and marine and aircraft engines. Engine design computations, and experiments on performance or development testing will occupy the laboratory periods.

Purchased for this course as well as for the automotive engineering work, a G-E Sprague electric dynamometer and a hydraulic dynamometer have recently been installed. The department has also obtained a new Hill diesel for laboratory work in its six courses in power machinery.

Professor Risteen, one of the fourteen faculty additions to the staff of Michigan Tech this year, comes with extensive laboratory experience. He has served in the aeronautical testing division of the Naval Aircraft factory at the Philadelphia Navy yard and as experimental engineer with the Comet Engine corporation of Madison. He will supplement instructional duties by continuing his research into high-temperature liquid cooling for aircraft engines, a subject on which he has published A. E. L. report 232. He will also study the effect of humidity on engine performance.

Given this fall for the first time at the Michigan College of Mining and Technology, the "Introduction to Architecture," a 12-week course presented by Associate Professor W. C. Polkinghorne, M.S. in C.E. (Yale) and E.M. (M. C. M. T.), enrolled 14 upperclass and graduate civil engineering students. The course consists of a survey of the principal architectural styles as developed through the ages, with especial reference to their present use. Illustrated lectures are given, and reference study and written criticisms called for. A two-term course, in which each section of the present work will be expanded, is planned for the near future.

Newark College of Engineering.—A single seater light sports airplane was recently presented to the Newark College of Engineering for use in its aeronautics course. The ship was designed and built by R. N. Thatcher of Montclair.

The craft is a monoplane with a twenty-five foot wing-spread and an overall length of eighteen feet. It is powered by a six cylinder imported Anzanni motor rated at sixty horsepower. The wing is of wood spar construction and the fuselage is of airplane steel with welded joints. The instrument board contains all of the indicators customarily used on the modern plane.

The craft will be used by the college to demonstrate methods of construction and bracing, and for aerodynamic and stress analysis.

Captain John A. Weishampel of the physics department of the Newark College of Engineering is the inventor of two devices that are of educational value in the laboratory.

One of these is an eccentric riveted joint rig. By means of this device the student first computes the load at which an eccentric joint will fail, and then actually tests the joint as a check. The apparatus is so arranged that a large number of rivet combinations may be used. The rivets are of special aluminum wire and are easily peened in the laboratory.

The second device is called an inertia and gyration pendulum. This apparatus is used to determine the plane moment of inertia and radius of gyration of beams, columns, or girders of any cross section, no matter how irregular the area may be. This means that many areas, whose moments of inertias were practically impossible to calculate, can be easily and quickly tested for their moment of inertia by using this instrument. It takes approximately twenty minutes to obtain a moment of inertia by this new method; whereas it took several days, if an irregular shape is being considered, to accomplish the same thing by the usual method. In the case of rolled sections, a thin slice sawed from the rolled shape, or a thin sixteen gauge pattern is all that is required.

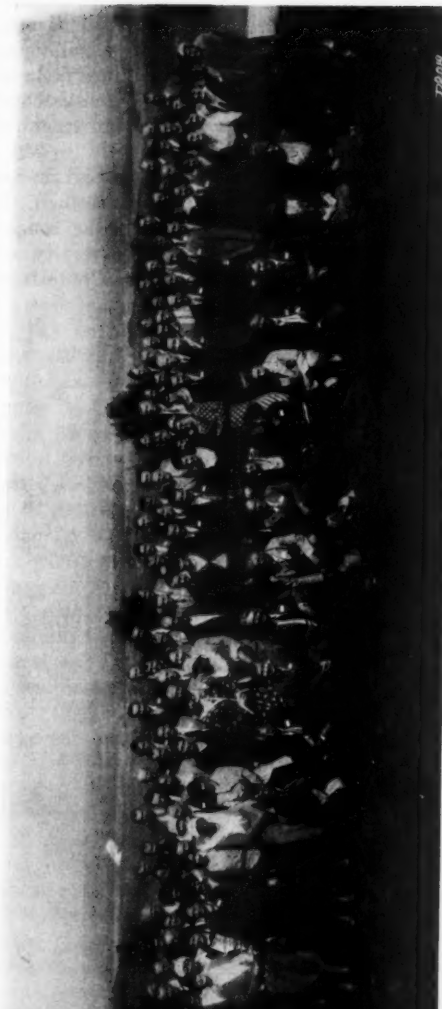
The simplicity of operation of the device adds greatly to its value. At present its percentage of variation, based on regular shapes such as circles and squares, is less than two and one half per cent. Refinements in its construction are expected to materially reduce the percentage of variation.

Northeastern University.—The faculty of the Day Division have voted to devote several monthly meetings during the current school year to a general consideration of tests and examinations. A special study will be made of the possibilities of new type objective quizzes in engineering courses. Dr. F. T. Spaulding of the Graduate School of Education at Harvard will give one or two informal addresses to the faculty this winter on the technique of teaching.

The faculty committee on student activities voted this fall to approve football as a varsity sport beginning in September, 1933

This action was taken in response to a persistent demand and increasing enthusiasm for football among the student body during the last ten years. Next year the University will sponsor only interclass contests with a limited number of intercollegiate freshman games, but the following year varsity football will be inaugurated on a schedule of not more than six games with New England Colleges.

Soccer will be dropped from the athletic program leaving the University with varsity teams in football, baseball, basketball, track and hockey. Announcement has also been made this fall that athletic relationships between Boston University and Northeastern University will be resumed this year. Contests will be arranged on a player control basis.



KANSAS-NEBRASKA SECTION OF S. P. E. E. MEETING AT MANHATTAN, KANSAS, OCTOBER 2–3, 1931

Kansas-Nebraska Section.—The Kansas State College members of the Society cordially welcomed the visitors from Lincoln and Lawrence at the Engineering Building on Friday afternoon, October second. After registering and renewing friendships the entire party drove to the polo field at Fort Riley where they enjoyed seeing a very interesting exhibition of fancy horseback riding given by officers of the army post. Following the exhibition the party returned to the Fort and as guests of the Kansas State College Faculty had supper in true army style.

The first session of the Section Meeting was called to order by President J. W. Haney at 7:45 P.M. in Recreation Center, Anderson Hall, with one hundred and fifteen present. President Haney appointed as a nominating committee Professor A. M. Ockerblad, University of Kansas, Chairman, Professor L. M. Jorgenson, Kansas State College, and Professor C. E. Mickey, University of Nebraska. These were to select nominees for the offices of President, Secretary and Chairman of the Program Committee for the ensuing year.

President Haney then introduced Dean H. S. Evans of Boulder, Colorado, national president of the Society. Dean Evans complimented the Section on the large attendance as being comparable to that of many of the national meetings. He stated that one of the most important functions of a meeting of this sort was to pass on to individual members the inspiration as well as helpful suggestions obtained in the meetings. The association with teachers from other schools and the discussion of their problems would be of great importance and value to those in attendance. President Haney in thanking Dean Evans said that he hoped that it would be possible to have the national president present at every meeting of this section. President Haney next introduced President F. D. Farrell, of Kansas State College, who spoke on the problem of liberalizing the education of technical men in the United States. This he said, was particularly difficult because of the trend toward specialization in this country. He pointed out that the knowledge of English was extremely important for the technical man because technical ability could not be clearly known unless the ability to talk or write were coupled with it. If engineers would reduce the time spent in reading sports news and the like and spend that time on substantial reading it would add greatly to their knowledge of general subjects. Professor Farrell in closing gave the members a very warm welcome to Kansas State College.

President Haney thanked President Farrell and introduced Mr. R. H. Fair, Plant Operating Supervisor of the Northwestern Bell Telephone Company, who read a paper entitled "Report Writing" by Mr. R. A. Gantt, Vice President of the Postal Telegraph and Cable Company of New York, who was unable to be here. The paper stressed the importance and necessity of clear, concise, and adequate reports in all phases of the engineering work. Inasmuch as the engineer spends a large proportion of his time in the preparation or study of reports, it is worthwhile for a young engineer to develop the ability of expressing himself logically and clearly along this line. A report should not be longer than necessary to convey the complete idea. If curves and sketches will clarify the matter, it is important that they be included in the report. At the present time there is a very apparent lack of training in report writing. This handicaps many young engineers.

A discussion of this paper brought out several additional points of interest. In the case of a long report a summary should be given. This should cover the proposition as a whole and refer to the conclusions. Dean Morgan of the Colorado School of Mines pointed out that the deficiency on the part of engineering students in the way of English frequently dates back to their earlier training. The Meeting then adjourned to Departmental Group Meetings.

Departmental Group Meetings

Agricultural Engineering: Leader, F. J. Zink

The topics brought up for discussion in this Meeting were as follows:

Engineering Training for Smith-Hughes Teachers
Elective Courses for Non-Agricultural Engineering Students
Laboratory Courses in Farm Motors
Background Courses for Agricultural Engineers

These topics were discussed in general with emphasis on the teaching of English.

English: Leader, F. N. Raymond

The need of a technical student for more intensive training in English was brought out. All teachers of technical subjects recognize this. Various plans used in different institutions were discussed. The cooperation of other departments with the English Department is needed in getting students to maintain a standard of good English in papers prepared under the direction of other departments. English courses should be required in the last two years also of the engineering course. This, it was pointed out, would lead the engineer to look upon English as an integral part of his college work.

Electrical Engineering: Leader, O. E. Edison

The question of laboratory reports occupied most of the discussion. Students complain that laboratory reports take more than their due share of time. Within the last few years the requirements have been somewhat reduced in the way of the amount of theory called for in the laboratory report. This has been reduced in many cases until the report deals only with the problem involved leaving the theoretical work entirely to classroom courses. The tendency of most undergraduate students is to minimize the importance of report writing. There were several graduate students present who, after a number of years of practical work having returned for graduate study, emphasized the value of the report writing which they had done in school. They stated that the training in report writing which they received in school had been of great value to them in their later experiences.

Mechanical Engineering: Leader, A. J. Mack

The problem of report writing was very freely discussed. The motion was made by Professor Luebs that the new Chairman of the Program Committee be requested to schedule departmental meetings again at the next annual meeting of the Section. This motion unanimously carried.

Applied Mechanics and Machine Design

The discussion centered around the writing of reports in connection with work done in the Testing Materials Laboratory.

Civil Engineering: Leader, P. F. Keim

In the discussion reference was made to the criticism of some engineers that they had not learned to express themselves clearly in college. This fault is not unique with engineers. The ability to think independently and to organize material effectively is another trait which some do not develop while in school. It was pointed out that if college graduates are not obtaining this, what must be the condition of the large numbers of our population that never go beyond the grades. The use of poor English in composition and conversation starts in the home or grade school. That is not entirely the fault of Engineering English instructors was brought out by Professor Kesner. There was also a discussion as to the criteria determining a modern school curriculum. It was pointed out that such a curriculum must represent a balanced selection of the common and specialized activities. The methods of teaching must be such as to teach the individual how to think rather than what to think. The outcome of such education must be weighted heavily

in terms of insight, appreciation and understanding rather than in skill of performance alone.

Saturday Morning

The Meeting was called to order in Recreation Center by President Haney at 8:30 A.M. with sixty-five present. President Haney in opening the meeting commented on the good attendance, which he attributed to the hospitality on the part of our hosts as well as a desire to see other instructors and compare problems.

President Haney introduced Professor W. C. McNown, University of Kansas. Professor McNown spoke on the subject of Curricula Flexibility. He pointed out that while some flexibility was desirable, the shortness of time for an engineering course made it necessary to stick to a rather well-defined series of subjects. The general objective of engineering education is to train men for strictly professional engineering practice. Training and discipline are superior to subject matter in education value. Most important just now is that the attention of Engineering educators be focussed upon the idea that they have made little concerted effort to formulate a definite educational philosophy to serve as a basis for guidance in connection with this and other questions of educational policy.

In discussion Professor P. F. Keim pointed out that a curriculum should above all things be well balanced as to subject matter. Professor H. A. Rice mentioned that too great flexibility would lead to having only a few men in each class. It would also require a greater number of instructors. Dean Shaad, of the University of Kansas, said that frequent changes in curricula would be unnecessary if the main fundamentals were included with only enough application courses to show the connection between theory and practice. Some choice outside of the main field followed by the students should be provided but too much variety tended to produce a smattering of information not particularly useful. Professor D. C. Jackson, Jr. commented on the problem of fitting students coming from Junior Colleges into the schedule of the University because of the fact that at the University a sequence of engineering subjects was started in the middle of the sophomore year. Professor O. E. Edison thought that sufficient flexibility should be provided to give the student at least some knowledge outside of his immediate line of work.

After a five minute recess the group reassembled and a paper entitled "Engineering Instruction," by Professor F. L. Brown, of the University of Kansas, was read by Professor A. M. Ockerblad. This has been called an age of specialization, although the urge for proclaiming oneself a specialist may have been carried to an ex-

treme in some quarters. Younger instructors are apt to expect too much of their students. Engineering instruction has not reached perfection. The existence of our Society is an evidence that we do not feel it has done so. The future may bring no revolutionary changes but rather a continuation of past development.

A large number of those present entered into the discussion of Professor Brown's paper. It was pointed out that the personality of the instructor has a great effect on the presentation of a subject before students. The policy of teaching should be to make students think rather than to tell them a large amount of detailed information. Time should not be spent for lecturing on material which the student can get for himself. The instructor should be sure of his subject in order to develop enthusiasm on the part of the student.

In the Business Session which followed, President Haney thanked Professor Hunt, Chairman of the Program Committee and Professors Raymond and Norris, members, for the extremely interesting program which they had provided. The minutes of the previous meeting held at Lawrence, Kansas, November 7–8, 1930, had been distributed to all as they registered and were approved by unanimous vote. Dean O. J. Ferguson invited the Section to meet in Lincoln next year. Dean Seaton moved that the next meeting be held at the University of Nebraska in Lincoln. The motion was seconded by Professor Russell. Motion carried. The report of the nominating committee by Professor Ockerblad, Chairman, was as follows:

For President: E. D. Hay, D. C. Jackson, Jr.
For Secretary: L. V. White, B. B. Brainard
For Chairman of the Program Committee: H. J. Kesner, L. A. Grone

It was moved by Dean Shaad that the first named for each office be declared elected. The motion was seconded and carried. Dean Shaad then expressed the appreciation of the visitors for the exceedingly kind hospitality they had received and moved a vote of thanks for the entertainment which had been provided by the Kansas State College membership. Dean Ferguson in seconding the motion was very appreciative of the hospitality which had been afforded. The motion carried unanimously. The meeting then adjourned to the Country Club.

The members met again at 12:30 at the Country Club. After having a group picture taken on the lawn they were guests of the Kansas State College Members at luncheon.

After luncheon President Haney introduced Dean George C. Shaad, University of Kansas, Dean O. J. Ferguson, University of Nebraska, Dean R. A. Seaton, Kansas State College, Dean J. R. Morgan, Colorado School of Mines, and Dean H. S. Evans, Uni-

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versity of Colorado. Dr. D. A. Worcester of the Education Department, University of Nebraska, and Dr. B. C. Hendricks of the Chemistry Department, University of Nebraska, were also presented. In a brief word Dean Evans emphasized the value of these sectional meetings in bringing instructors of neighboring schools together in a most friendly way to talk over their problems. Engineers must take a more prominent part in the solution of industrial problems in the future. The engineer has a responsibility as well as an opportunity in being a guide to industry.

On behalf of the visiting ladies Mrs. J. W. Haney thanked the Kansas State College faculty ladies for their generous entertainment. President Haney again expressed his appreciation for the record breaking attendance. He introduced Professors Hunt, Raymond and Norris and thanked them for providing an unusually interesting program. The new officers were next introduced:

President: E. D. Hay
Secretary: L. V. White
Chairman of Program Committee: H. J. Kesner

Those present were reminded of the meeting next fall to be held at Lincoln, Nebraska. All were urged to be there.

Final adjournment was at 1:30 P.M.

R. W. WARNER,
Secretary.

The annual meeting of the **Middle Atlantic Section** was held at Lafayette College, Easton, Pa., Saturday, October 31, 1931. A total of one hundred and twenty-nine members and guests were in attendance—representing eighteen different education institutions and ten firms.

A number of delegates visited the Lehigh Portland Cement Company plant in the morning. The headquarters was in the new John Markle Mining Engineering Hall, which was inspected by another group of delegates. A complimentary luncheon was served to approximately one hundred and twenty-five delegates and guests in the Gymnasium.

Chairman S. S. Edmands, of Pratt Institute, called the meeting to order at 2 P.M in Pardee Hall Auditorium. Dr. William Shafer Hall, Clerk of the Faculty and Professor of Mathematics at Lafayette College, welcomed the assembly and spoke briefly of the history of engineering at Lafayette. The minutes of the spring meeting held at Pratt Institute, May 9, 1931, were read and approved. The treasurer reported a balance of $119.51, with some small bills for printing and postage outstanding.

The following papers were then presented:

"The 1931 Stevens Institute Preparatory School Camp," Harvey N. Davis, President of Stevens Institute of Technology.

"Electrons at Work and Play." Lecture Demonstration by Dr. Phillips Thomas, research engineer for the Westinghouse Company.

"A Comparison of Engineering Education in Germany and the United States," Dr. Ernst Weber, Research Professor of Electrical Engineering at Brooklyn Polytechnic Institute.

"Some Aspects of Engineering Education," Robert E. Doherty, Professor of Electrical Engineering at Yale University.

There was considerable discussion of these very interesting papers and the chairman appointed committees on resolutions, nominations and meetings.

Following a brief inspection of the buildings and grounds of Lafayette College, dinner was served in the Gymnasium at 6:30 P.M. Professor William Mather Lewis of Lafayette College gave a most interesting and inspiring address.

The Committee on Nominations, consisting of Dean Arthur M. Greene of Princeton, O. W. Eshbach of the American Telephone and Telegraph Company, and George A. Stetson of the American Society of Mechanical Engineers, made the following nominations for the ensuing year:

For chairman—Dean Robert C. Disque of Drexel Institute.

For secretary-treasurer—Charles G. Thatcher of Swarthmore College. This report was accepted and the nominees unanimously elected.

The Committee on Meetings, consisting of E. H. Rockwell of Lafayette, F. E. Mehrhof of Rutgers, D. B. Porter of New York University, and C. E. Davies of the American Society of Mechanical Engineers, recommended that the Section accept the invitation of Newark College of Engineering for the fall meeting, 1932, and the invitation of the University of Delaware for the spring of 1933. This report was unanimously accepted.

The Committee on Resolutions reported as follows:

"The Middle Atlantic Section of the Society for the Promotion of Engineering Education has enjoyed in full measure the hospitality of Lafayette College in connection with the Fall Meeting of October 31, 1931. We are impressed by the beautiful site of the college, by the adequacy and dignity of its buildings and by a spirit of idealism and culture that seems characteristic of its atmosphere. The arrangements for the conduct of our meeting have been most satisfactory, and the courtesy of the College in providing a generous luncheon and in making possible our delightful dinner of the evening is heartily appreciated. For these and many other attentions we cordially thank the College, its officers, and all others who have contributed to this happy result.

"In particular we express our congratulations to Professor Morland King for representing so effectively in his own person the hospitable spirit of his institution.

"The courtesy of the Lehigh Portland Cement Company in showing us its plant is gratefully acknowledged.

"Our cordial thanks are also hereby extended to the Westinghouse Electric and Manufacturing Company, and to Dr. Phillips Thomas, for a most instructive and entertaining demonstration lecture on 'Electrons at Work and Play,' to our visitor from abroad, Dr. Ernst Weber, and to those of our own members who contributed to the program of the afternoon.

"We shall leave Easton with a renewed interest in the College that has provided for us so successful a meeting."

(Signed)

HARVEY N. DAVIS, Stevens Institute,
FRANK L. EIDMANN, Columbia University,
GEORGE F. BATEMAN, Cooper Union.

This report was unanimously adopted.

The Section then adjourned to meet again at Swarthmore in the spring of 1932.

CHARLES G. THATCHER, *Secretary.*

A meeting of the **Newark College of Engineering Branch** was held on October 29, 1931. The purpose was to discuss the frequency and subject matter of future meetings. It was decided that the Section would meet every two months. President Cullimore requested that the meeting to be held in December be turned over to him. In view of the fact that the method of teaching English in engineering colleges has undergone more or less radical changes and since such a change has taken place at the College of Engineering, it was proposed that Dr. Giesy, the head of the English Department, be invited to discuss the matter at the February meeting.

The Section voted to extend an invitation to the Middle Atlantic Section of the Society for the Promotion of Engineering Education to hold the Fall Meeting of 1932 at the Newark College of Engineering.

The fall meeting of the **New England Section** was held at Brown University on Saturday, October 10. Total registration was seventy-nine. The representation of teachers in New England was as follows: Massachusetts Institute of Technology, twenty-one; Brown University, fourteen; University of Maine, eleven; Tufts College, eight; Worcester Polytechic Institute, six; Harvard University, three; Yale University, two; Norwich University, two; Northeastern University, two; Connecticut Agricultural College, two; Dartmouth, one; University of New Hampshire, one; Trinity College, one; Wentworth Institute, one.

After a brief assembly in the morning the group inspected the facilities of Brown University, following which they were guests of the University at luncheon at Faunce House, the new student union.

The afternoon session began with a welcome to delegates by Dr. A. D. Mead, Acting President of Brown University. The main topic of the meeting, "Engineering Education in Light of the A. S. M. E. Report on the Economic Status of the Engineer," was then taken up. Dr. C. F. Hirshfeld, Chief of the Research Division of the Detroit Edison Company, member of the committee making the investigation and former professor of mechanical engineering at Cornell, gave an excellent summary of the report, pointing out primarily the great difference in earnings between those engineers having executive responsibilities and those whose work is purely technical. Dr. Harvey N. Davis, President of Stevens Institute of Technology, then discussed engineering education in light of this report. The addresses were followed by a spirited discussion in which a large number of those present took part.

A considerable number of wives of members were present for the meeting and were entertained by a local ladies' committee during the afternoon's technical session. They joined their husbands, however for the evening dinner at the Brown Faculty Club, where all were later addressed by Professor Will S. Taylor, head of the Art Department at Brown University, who took as his topic "Constructions which Artists Use." He illustrated his points by a large number of crayon sketches and by lantern slides.

Meetings were presided over by the chairman, Paul Cloke, Dean of Engineering at the University of Maine. The local arrangements were in charge of Professor James A. Hall of Brown University, secretary of the New England Section. For the coming year Raymond Hall was elected chairman of the Section and Raymond R. Marsden, Dean of the Thayer School of Civil Engineering at Dartmouth, was elected secretary.

The T-SQUARE PAGE

DEVOTED TO THE INTERESTS OF THE DIVISION OF ENGINEERING DRAWING

FREDERIC G. HIGBEE, EDITOR

An Annual Competition in Engineering Drawing. For a number of years the Division of Engineering Drawing has considered the desirability of an annual competition concluding with an exhibition at the annual meeting. A committee is now preparing a plan to be submitted for approval at the 1932 meeting. In order to secure data and to stimulate interest each department of engineering drawing is invited to submit to the editor before June 15, 1932, twelve blue prints of the best tracing made in the first course in engineering drawing during the year 1931–1932. One print from each department will be exhibited at the annual meeting, judged by a committee, and awarded honors. The remaining prints will be circulated as a traveling exhibit during the following year. Suggestions and ideas concerning the competition will be welcomed by the committee on competition.

Engineering Drawing Problem. How many shapes may the object have which is shown on page 89, Figure 172, French's Engineering Drawing? To simplify solutions consider only the small part which extends above the large block, and consider only types and omit variations of the same type.

Readings for Teachers of Engineering Drawing. Contributed by Donald R. Dohner, Engineering Art Director, Westinghouse Electric and Manufacturing Company: "Windowless Factory"—Architecture—May, 1931, pages 303 to 304; Art of the Machine"—Architecture—May, 1931, pages 273 and 376; "Art Modern Design is Inevitable," by Edward Richardson—Toilet Requisites—May, 1931, pages 26, 27, and 52; "Color Schemes for Four Bathrooms," by Elinor Raymond—House Beautiful—May, 1931, pages 46 and 47; "How Design Applies to Lighting Fixtures," by Thomas L. Dusseau—Lighting—April, 1931, pages 18, 19, and 27; "Beauty and the Balance Sheet," by Abbot Kimbal—Advertising Arts—May, 1931, pages 15 to 22; "Cigarette Package of the Future," by Egmont Arens—Advertising Arts—May, 1931, pages 28 to 32; "Kitchen Sink Goes Modern"—Metal Work—May, 1931, page 297; "Light and Color of Chromium," by Carleton Tyder—Good Furniture—May, 1931, pages 244 and 247.